Edukacja. An interdisciplinary approach 2017, 72–83
doi: 10.24131/3724.170505
ISSN 0239-6858

Using various representations in the process of solving mathematical problems

Ewa Swoboda
Faculty of Economics, University of Rzeszów*

Mathematics uses a wide range of representations, but the mathematical symbol is not the only way to code information. Different ways of representing mathematical concepts and relationships are used, especially in the early stages of learning. Generally, the teacher decides on the choice of representational forms to use. But in the process of solving mathematical problems, it is the pupils – not the teacher – who are engaged in the problem-solving, and the coding used should support their cognitive work. This paper analyses how different representations can influence the results of work on an untypical mathematical problem. The task was solved by a group of 7–8 year-old pupils participating in a mathematics club. The examples selected for analysis indicate a strong relationship between the choice of representations and the final result of the pupils' work.

Keywords: mathematics; problem solving; mathematical symbols; enactive representation; iconic representation.

Mathematics is perceived as a science that employs abstract concepts and relations. Indeed, it is not possible to empirically experience what a specific mathematical concept is, or what the connections and relationships between the concepts are based on. These objects and relationships can only be represented by a word, picture, symbol, gesture. Each such representation carries a specific content, is a code for a certain meaning. Regardless of its external form, representation should render a specific mathematically determined sense. Reading it can be compared to the ability to use language, which must be learnt. Therefore, learning mathematics must go hand in hand with learning the language that mathematics uses. Adoption of such an assumption naturally opens up a broad field of research for teachers, historians of mathematics, linguists. Linguists who study the language of mathematics analyse the phenomenon from very different points of view. Ladislav Kvasz (2014, p. 207) claims:

Mathematics is usually understood as the language of science. It is perceived as a tool with which disciplines like physics or economics acquire their precision. Therefore, it is often overlooked that mathematics itself has its linguistic dimension: the same mathematical content can be expressed in many different ways.

In education research, a broad spectrum of "languages" is analysed: mainly spoken language (mathematical and general), non-verbal language, symbolic language, figures and graphs, quasi-mathematical language (Guidoni, Iannece and Tortora, 2005; Pirie, 1998; Slezáková-Kratochvílová and Swoboda,

* Address: al. Rejtana 16c, 35-959 Rzeszów, Poland.
E-mail: eswoboda@ur.edu.pl

2006). Researchers in mathematics education look at language with a specific purpose in mind. They examine its connections to the thinking processes occurring while learning, doing mathematics. All forms of written language can be a source of this kind of knowledge. Independent pupils' writing can inform the teacher about his or her thought processes. Candia Morgan (1998, p. 33) writes:

> The assumption of a strict connection between thought and language is an indispensable element in such research, which aims at using a text (written or oral) created by students as evidence of their thinking.

Obviously, in the reality of school, this natural process may be corrected in different ways. Created records are often influenced by the social contracts in place at school (concerning, e.g. the way the solution of a problem is written down), the preferences and skills of the pupil. However, both constructivists and academics referring to socio-cultural theories agree that knowledge is a subjective reflection of prior experience, it is a subjective attribute. If we assume that language is the expression of thoughts (Wygotski, 1989), and that "language develops in accordance with thinking habits" (Piaget 1992, p. 144), all records, statements, codes and signs used in the area of mathematics should be closely connected to thinking and should support individual solutions and strategies.

Theoretical grounds for the study

It is wrong to equate ways of presenting mathematics only with the use of mathematical symbols, irrespective of the fact that the symbol does indeed play an important role. The history of mathematics shows that the introduction of many symbols greatly simplified the record (Ifrah, 1990; Struik, 1963) and facilitated reasoning. Currently, it is even sometimes emphasised that the mathematical symbol "thinks" for the mathematician (Krygowska, 1977). In school teaching, the period of building connections between the sign and the object (situation) it represents should be long. This is the only way in which the informative function of the sign and symbol can be familiarised, and its usefulness can be guaranteed.

> […] the sign will always maintain a connection with the practical activity of an individual and [we should] think of the sign as a semiotic object functioning in the environment, in which the specificity of a particular activity is considered (Radford, 2003, p. 50).

This is why mathematics cannot be reduced to a system of arbitrarily supplied signs and symbols, nor even to formally introduced concepts and relationships. Heinz Steinbring (2005, p. 19) expresses this idea in the following way:

> Mathematical concepts represent relatively autonomic epistemological units. In order to make students understand a mathematical concept, […] they must be active in that cultural environment and must detect the possible interpretation of the mathematical concept. Mathematics should be understood as an activity.

The activity underlined by Steinbring is connected to the essence of mathematical operations at each stage of carrying them out. This activity requires assigning one's own meaning to concepts, the relations between concepts, as well as the symbols that encode concepts and relationships. The ability to appropriately use different forms of representation can be crucial. Without doubt, such differentiation is very helpful, especially at the early stages of mathematical education. The breadth and freedom in using different representations make them useful tools, rather than obstacles.

Analysing the development of reasoning, it is worthwhile to utilise some of the findings of education experts (Pirie and Kieren, 1989). They draw attention to the "recursion" of mathematical understanding – the need to continuously return to earlier levels of understanding and the ability to look at them from a higher level. Children uses their inner, intuitive knowledge in building more complex knowledge. This provides the opportunity to integrate different fragments of knowledge, experience mathematical facts more deeply and understand language better. Figure 1 presents a fragment of a diagram which shows the connections between the different levels.

The overlapping loops indicate that an action can have different functions at different stages of reasoning. Only after achieving a certain level of familiarity with the material is one able to notice non-obvious properties, while the formalisation of empirical facts may provoke further acting in order to observe the inferred dependencies. One of the associations suggested by the diagram can lead towards a reference to the spiral arrangement of content, known from education studies. The skills and experience acquired at lower levels should enable understanding them at the higher level. Bits of information should, therefore, be obtained in a manner that binds them into a structure, and facts should be connected in accordance with some rules and general concepts, from which they can be derived (Filipiak, 2008).

The diagram above can also be referred to the well-known findings of activity-based teaching of mathematics (Krygowska, 1977; Siwek, 1988). The actions can be interpreted as actual activities (at the first stage), imagined ones (at a higher level) and mental ones (the highest). In the tradition of school teaching, a hierarchical structure has been established, which suggests that the higher educational levels should not divert from the actions involving actual objects, substituting them with mental manipulations. It seems that such interpretation is incorrect and probably inconsistent with the intentions of the creators of activity-based teaching of mathematics. This narrow treatment is also

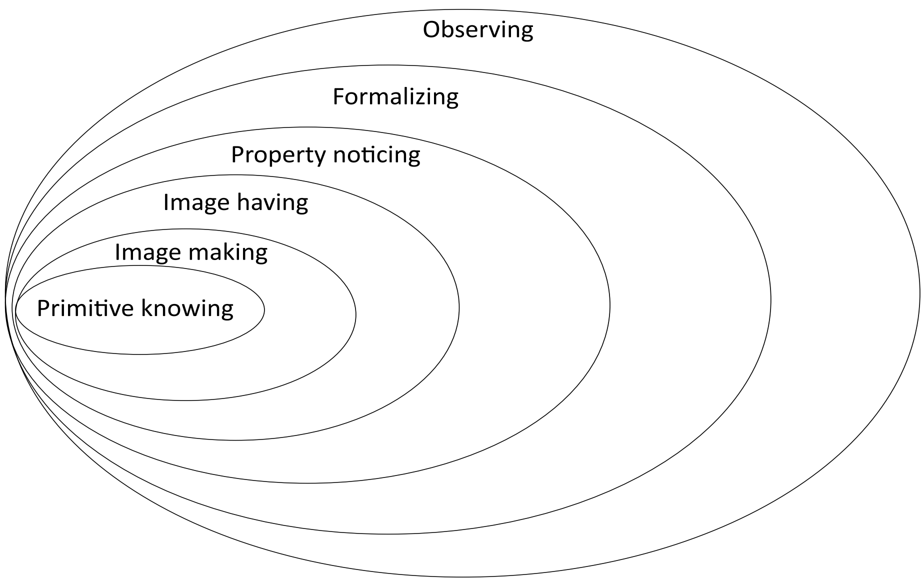

Figure 1. Recursive model of development of mathematical knowledge.

Source: Pirie i Kieran (1989, p. 7).

contradicted by the results from other fields of science. The need for different forms of support for abstract thinking is confirmed by research carried out in the field of neuropsychology, or even the philosophy of mathematics (Brożek and Hohol, 2017). Krzysztof Cipora, Monika Szczygieł and Mateusz Hohol (2014, p. 62) point to the "connections between the mental representation of fingers and the counting skill" and confirm that the performance of some gestures with fingers to represent numerical values relieves working memory and enables more comprehensive mathematical cognition. The authors, referring to broad studies from other countries, state that "Both at the behavioural and the neuronal levels, processing numbers is linked to the activation of the mental representation of fingers" (p. 63). Fingers are the basic, most obvious set of counters that a child has access to. But the very representation of values on fingers and using the kinaesthetic sense is insufficient. What is needed is support from the verbal code, the use of the visual recognition of patterns. This is how Albert Einstein referred to the mental manipulation of abstract objects:

> Words and language, whether spoken or written, do not play any role in my thought process. The mental blocks which are used to construct my thoughts are some signs or images, more or less clear, which I can freely recall and recombine (in Brożek and Hohol, 2017, p. 193)[1].

This theoretical approach will be used in analysing the work of a selected group of eight-year-old pupils. The emphasis was placed on various forms of representation, in which symbol, word, picture, and gesture influence the outcome of the work to varying degrees.

[1] According to Bartosz Brożek and Mateusz Hohol, this quotation is from a letter from Albert Einstein to Jacques Hadamar (in Ghiselin, B. (ed.). (1980). *The creative process: reflections on invention in the arts and sciences* (pp. 43–44). Los Angeles: University of California Press.

Organisation of observations, research tool, methodology

The aim of the observation was to look for an answer to the following question: What is the function of different way of encoding information in the process of solving problems? Detailed study questions were as follows:

- How is the verbal information that describes a quasi-mathematical problem read by pupils?
- How can pupils represent the information provided in the problem?
- What form of representation of a problem will turn out to be the most advantageous for successfully finding a solution to the problem?
- Are there any forms of representing a problem that might constitute an obstacle in carrying out the reasoning, in finding the answer to the formulated problem?

The following problem was used as the research tool:

> 7 squirrels decided to have a race. 31 nuts were to be used as awards. The owl decided that the fastest will receive more than the slowest, and additionally each squirrel should receive a different number of nuts. How can this be done? How many solutions are there? (Baggett and Erhenfeucht, 1998).

The problem was presented verbally, that is, as a "word problem". The only mathematical symbols that existed in the text concerned the quantity and they were: the number of race participants and the value of the award. All other relations needed to be decoded by the pupil and in this case, it was difficult to look for a keyword which would guide them to known mathematical relations. The story itself was a veil for a mathematical problem formed by the decomposition of number 31 into a sum of seven different elements. In addition, reference to customs relating to competitions, in which the better competitor receives more and the worse one less, imposed a descending (or ascending) order of elements. Although this condition was not unambiguously written down, prior experiences of working with this tool showed that children interpret it in just this way. Shortening the text facilitated their memorising the conditions presented in the problem.

This problem can be very easy if solved at the level of representation with the use of counters by applying an appropriate manipulation. It is sufficient to somehow represent seven race participants (seven buttons, seven pieces of paper). Then, divide the counted 31 chips in turn, e.g. one for the first one, two for the second one, three for the third one… or: one for each of them, then adding another to those from the second to the seventh one, then another one for those from the third to the seventh one… In this way 1 + 2 + 3 + … + + 7 = 28 chips will be distributed. The remaining chips can: either all three be given to the last one, or two for the last and one for the last but one, or one for each of the last three. These are all possible solutions – each race participant has a different number of nuts, whereas the principle of the fair distribution of awards is maintained.

Such real manipulation can be replaced with an imagined manipulation, supported directly with a symbolic script. Then, a slightly different reflection on the sentence will be needed. Writing 1 + 2 + + 3 + … + 7 can reflect the mental division of awards among seven race participants, maintaining the condition of unequal numbers. Calculating the value of the sum (finding the number 28) will provide the basis for further mental operations, which will be manifested in increasing the value of some elements of the previously recorded sum and observing the remaining relations between the elements.

The problem was tackled by 17 pupils aged 7–8 in a mathematics club at one of the primary schools in Prague. The classes were held in a classroom equipped with tables, and while working on the problem, the children sat on the carpet. They could move around freely, work in a group or separately, share their ideas. The author of this article was present in the room as an observer. The whole class was filmed and the collected material (video, children's worksheets, photos taken in the course of the class) was the basis for further analysis. The problem was first read aloud by the person conducting the class. Then, pupils received the text of the problem on a sheet of paper, on which they could also write their solutions. In addition, they could use counters (beans), which the person conducting the class spilled from a big bag. The course of their work will be used to illustrate the conclusion that the choice of various forms of representation may have varying impact on the thought processes relating to problem solving.

Results of the observations

The problem was new to the children. They knew that it was a mathematical problem and as such, they thought that it should be solved in a traditional way, that is, with the use of mathematical symbols. However, it was obvious that they did not know how to use the sheet of paper to write solutions or check the reasoning. Sometimes they made chaotic attempts at counting "something", which can be seen in Photo 1.

However, even when one pupil had a specific idea that could lead to finding a solution, he/ she was not always able to reconcile it with formal notation. The habit of representing a problem with symbolic notation turned out to be a significant obstacle.

Symbol

One of the boys started by writing down (the divided nuts) by trial and error. He

Photo 1. Attempts at using traditional mathematical code to solve the problem.

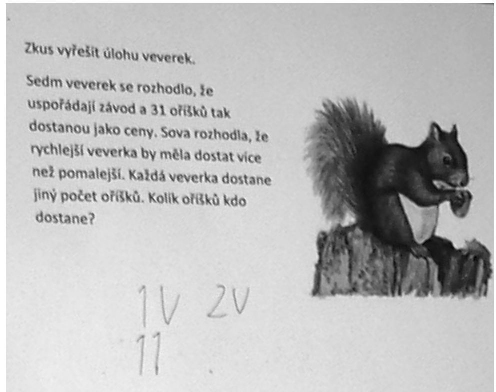

Photo 2. The first boy begins working on the squirrel problem.

wrote "1ve, 2ve" on the paper (labelling the first and second squirrel in this way), under which he wrote 11, 10 (Photo 2). Proceeding in this way, he created both his own notation and used traditional mathematical symbols. In this way, he made use of 21 nuts. Then, an equation appeared: 21 – 9 = 12, meaning, in a possible interpretation, that the boy wanted to assign 9 nuts to the subsequent squirrel, and at the same time assess, how many nuts would be left to be divided. Such notation, however, uses the relations between numbers in the wrong way. This stage of work shows that the pupil did not know how to formally write all the relations he saw in the problem.

Apparently, he was not satisfied with the result, as he crossed it out and started working with other numbers. The subsequent notations were partial, e.g.: 5, = 20 (written from top to bottom) and at this stage only a "prop" for thought processes, which were probably still chaotic. Since he noticed that other children were using beans, he also reached for them. During the manipulation, he worked with a friend. The number of beans they were working with did not agree with the quantity given in the problem. Through manipulation, they managed to divide the beans into seven groups of increasing quantity. Asked by the person conducting the class to write his solution, he continued working out the formula that was already written on the paper. To the existing labels "1ve, 2ve", he added: "3ve, 4ve, 5ve, 6ve, 7ve" and the numbers: 13, 7, 6, 5, 3, 2, 1 under them. He wrote these numbers while counting the beans that lay before him (Photos 3 and 4).

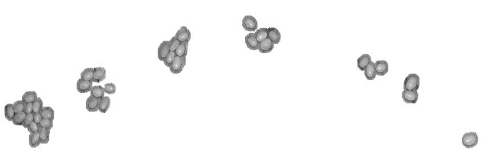

Photo 3. Division of the beans into 7 unequal sets.

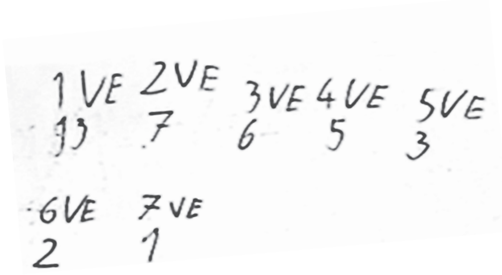

Photo 4. Notation of the solution developed by the first boy.

Commentary. The boy experienced failures while solving the problem in a traditional way, using typical mathematical notation, although his initial actions were correct. He looked for solutions, sensibly making mental estimations, used these estimations in his notations, and made partial calculations. He tried to give his writing a mathematical form, using both a certain form of a table (labelling specific squirrels and attributing estimated values to them), and a mathematical record of the operation. The later notation of the solution was partial and served different purposes. Transitioning to manipulations enabled the boy to see that finding a solution could be easy. After dividing the beans into seven sets, he did not try to make any more notations. He did not feel the need to write down the obtained result. One could venture to say that he did not wish to return to the form, which in his opinion, did not help him find a solution. Provoked to write something down, he did so quickly without verifying his result with the initial information provided in the problem.

Picture

A group of three girls working together showed no interest in the counters. When they received the paper, on which the text of the problem took up only a small fragment, they decided to illustrate the text with a picture. They started working by drawing a podium for the winners. The very process of drawing made them forget about the mathematical aspect of the task and they focused on the realistic problem of rewarding the race contestants. One can say that that the piece of verbal information was most important for them and pushed all other information into the background. The girls probably used a model known from observation, for the drawn podium was composed of three numbered steps (Photo 5). Squirrels were drawn on the steps. Above the podium, they wrote "31 nuts". The value 31 was divided into three components: 10, 15, 6 (the drawing suggested three winners), and the numbers were written above the marked places for subsequent winners. To avoid any doubt about how many nuts were to be given to particular squirrels, arrows were added to the numbers to indicated the places on the podium – the largest of the numbers was attributed to the winner. Thus, the picture became a graphic representation of the awarding procedure.

The next photo (6) was taken after a discussion organised during a break with all class participants. Then, the girls concluded by themselves that they had misinterpreted the problem. Regardless of this fact, they decided to keep using only a graphic

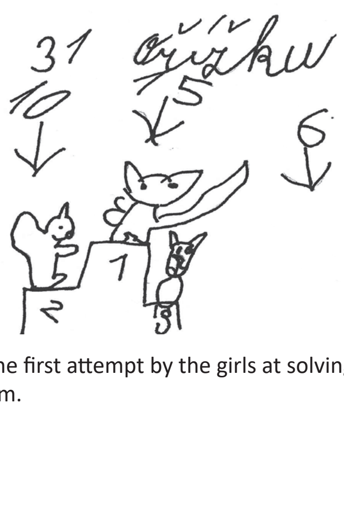

Photo 5. The first attempt by the girls at solving the problem.

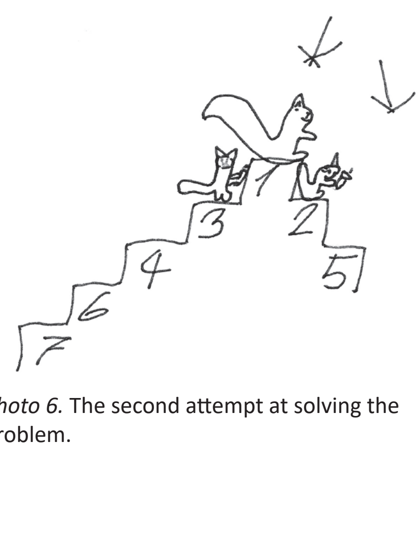

Photo 6. The second attempt at solving the problem.

representation. The drawing is unfinished, which documents the stages of its development in an excellent way. The podium has seven steps with numbered places, with squirrels standing on the first three. This is an extended version of the first drawing. There are also arrows indicating first and second place on the podium. It seems that this is preparation for later work, which did not take place. Most of all, however, no reference is made to the numerical values, to attempts of dividing the pool of awards.

Commentary. The girls used a clean sheet of paper not to solve the problem, but to illustrate the results of the competition. They worked as a team, the picture was also the result of their joint work. The literary content of the problem (race, competition, rewarding competitors) dominated the mathematical content. Their realistic illustration, referenced to their life experiences, overshadowed the later course of work. Changing the conditions by limiting the number of winners to three competitors led to a situation in which the mathematical problem became easy. The girls did not feel the need to encode the way the solution was obtained, they only wrote down the result. The next stage, relating to the correction of the representation of the problem, took place only at a graphic level. It did not, however, influence the strategy of working out the solution. Perhaps the girls still intended to attribute awards to subsequent competitors (arrows!), but now the task was no longer easy.

This group did not finish their work, they may have run out of time – the later part was dominated by a joint discussion and attempts at finding a solution by means of "drama".

Gestures (manipulation)

Another boy started by selecting 31 beans and placing them on a sheet of paper – in this way, he marked the area of his work. He started by dividing them into seven sets, apparently estimating that he had many elements, and then added a few to each set. Through manipulation, he managed to divide the beans into unequal sets, yet not maintaining order with regard to the subsequent quantities. The only thing he worked on next was arranging the beans into "chimneys" (Photos 7 and 8).

Perhaps this layout was supposed to help him assess (arranged according to quantity) this division. However, he did not decide on a different arrangement, so not all of the chimneys are elongated. Apparently, he concluded that such a layout provided him with sufficient information to find the solution. The manipulation stage was for him the initial, most important step in solving the problem.

Then another, mental process took place – encoding the obtained solution with the simultaneous arrangement of the results according to the information provided in the problem (Photo 9). This is just the "report",

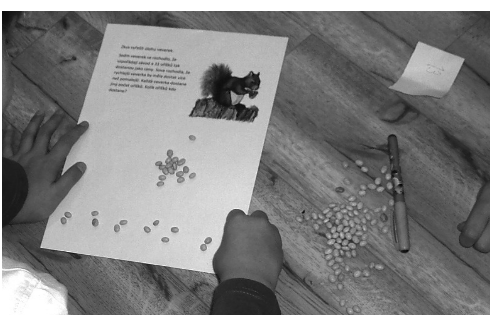

Photo 7. Selection of 31 counters and dividing them into seven sets.

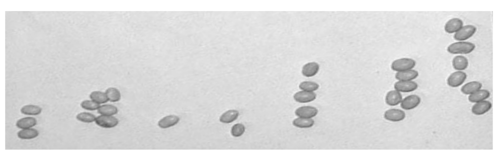

Photo 8. An attempt at differentiating the numbers of elements in the sets.

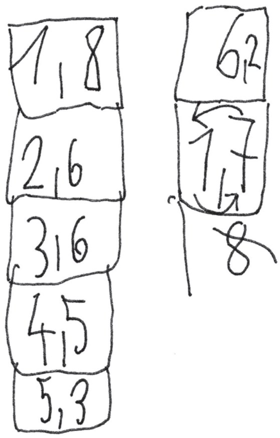

Photo 9. Report on solving the problem submitted by the second boy.

as nothing was crossed out on the paper, and the sum of digits indicating the value of each award reaches 31, which is consistent with the numbers provided in the problem. His solution presents a sequence of squares, like attached windows. The sequence runs from the top of the sheet to the bottom, and when he ran out of space – on the right-hand side. One can see that rewriting the solution onto paper entails subsequent intellectual work, as it is arranged differently than the layout of the beans. In each window, the boy wrote two digits separated with a comma: the place of the competitor and the value of the award. Thus, two number orders were made – one ascending (subsequent places of the competitors), and the other descending (values of the awards). With this record, the signs from the first window (1, 8) should be interpreted as: the winner shall receive 8 nuts as an award. Surprisingly, he did not notice that he had assigned the same number of nuts to squirrels from second and third place, which could be explained by the focus of attention on appropriately encoding the obtained solution. Due to running out of space on the sheet, the boy had to draw more windows – which distracted him. It may be that he first wrote the sequence of digits: 6, 7, 8 (the last crossed out), and only then assigned the value of the award. In the last window, he mistook the order of the symbols for the place of the competitor and the value of the award, but controlled the whole notation and corrected it by using arrows to mark the order in which the digits should be read.

Commentary. The pupil decided to solve the problem through manipulation. In a conscious manner, he complied with the conditions of the problem: he selected 31 objects for manipulation, identified the seven positions of the competitors, attempted to differentiate the values of the awards. This way of working led him to an almost complete and correct solution. The notation on the paper represented for him a way of reporting what he had done earlier. Yet it can be clearly seen that the work at the second stage required a completely different approach. Forced to write down the solution, he started a totally different thought process. Nevertheless, the initial, manipulation phase of the work made it possible for him to continue the work mentally. Now one can see that the boy understood the sense of the digital notation, could apply the appropriate symbols, encode a digit both in the cardinal aspect (the number of nuts as an award), and ordinal aspect (subsequent place taken by a competitor). In addition, he tried to maintain the order (ascending and descending) among the elements.

Findings

All representations should be a reflection of thought processes, and those – within mathematics – should concern functioning in mathematical models. The examples presented in the quest to solve the problem show that pupils who do not have to follow imposed forms of writing the solution often use their own ways of representing mathematical content. These in turn can help in varying degrees to achieve success.

Activities, appropriate gestures, actions are the easiest form of expressing content, and additionally, they seem to be the closest to the spontaneous behaviour of pupils. It is worth remembering that the application of manipulation should not be treated as “moving to a lower level”, which may be considered by more able pupils as a kind of degradation, undervaluing their intellectual abilities. The appropriate construction of a physical model and transforming it enables dynamics and action to be introduced to reasoning – often overlooked in symbolic notation. Changing over to symbolic encoding may turn out to be much easier, and the symbols themselves may refer only to the mathematical aspects of the encoded phenomenon.

Drawing has a special place in the representation of mathematical concepts. The discussed example testifies to the fact that it cannot be just any drawing, loosely connected with the topic at hand. Focusing on details may distract from the essence of the problem. Also, connecting too closely with one’s own experience may hinder the perception of new data relating specifically to the situation depicted in the problem. Skilful extraction of mathematical relations through an independently drawn picture may be very helpful in understanding a problem (Dąbrowski, 2006; Nowakowska, Orzechowska, Sosulska and Zambrowska, 2014). This skill, however, is not automatic, which is confirmed by education studies (Reclik, 2012; 2015; Stankiewicz, 2016). The iconic representation of mathematical problems is not at all easy. It is related to acts of abstraction – many irrelevant details must be discarded and additionally, important mathematical relations must be displayed in such a way that specific relations become obvious. Attention should be paid to these facts when analysing textbooks prepared for the stages of early education, which are often overloaded with pictures that not always have a mathematical purpose. On the other hand – it is worthwhile to support pupils in the independent creation of a “mathematical” drawing.

The discussed solutions were developed by pupils working in one group. They were free to choose any interpretation, to use the counters, to use the worksheet in whichever way they liked. As the pupils were not selected in any particular way, we can suppose that similar behaviours could take place in any typical school classroom.

The presented solutions do not exhaust all that were developed. Such diversity provides enormous opportunities to conduct discussions in the class. At the discussion stage, the problem itself may only be a pretext for building mathematical knowledge. Many pupils acted by manipulating the beans – in the course of the discussion, they could talk about this, verbally describing their actions, explaining their purpose. Others encoded the results symbolically. A report on such encoding can be a pretext for taking a new look at prior actions. Also, notations using mathematical symbols, the arithmetical record of operations, should be introduced, that is, there should be a change over to the formalisation of the observed relations. Such records are a different form of both the prior manipulations, as well as the encoded solutions.

Conclusions

When reasoning mathematically, children refer to their personal experience. The subjective interpretation of the provided information (problem) allows them to take the first step in finding the solution. While working on a problem, pupils can support their reasoning with different types of representations, in accordance with their way of encoding, using informal language, gestures.

The form of representation supporting the process of finding the solution can differ from child to child. Children should have the opportunity of choosing how to represent both the problem, as well as the way of solving it.

Using mathematical notation is not easy. Like every sign, a mathematical symbol is a carrier of specific meaning. It should be associated with different procedures, with typical applications. However, such associations are built over a long time by collecting experiences. These facts are frequently emphasised by mathematics educators:

> Mathematics deals *per se* with signs, symbols, symbolic connections, abstract diagram and relations. The use of symbols in the culture of mathematics teaching is constituted in a specific way, giving social and communicative meaning to letters, signs and diagrams during the course of ritualised procedures of negotiation (Steinbring 1997, p. 50).

Although mathematics has its symbolic notation, recording mathematical operations only by means of mathematical symbols, even from the earliest stages of education, may be a significant barrier for children (Rożek, 2016).

The diversity of representation greatly enriches opportunities for using mathematics. Applying different notations and using different forms of mathematical expression provides the opportunity for the most adequate representation of mathematical thought to be developed for the given situation. However, each representation – from an educational perspective – entails other problems, of which the teacher should be aware.

Literature

Baggett, P. and Ehrenfeucht, A. (1998). *Breaking away from the math book II: more creative projects for grades K-8*. Lancaster: Technomic Publishing Company.

Brożek, B. and Hohol, M. (2017). *Umysł matematyczny.* Kraków: Copernicus Center Press.

Cipora, K., Szczygieł, M. and Hohol, M. (2014). Palce, które liczą: znaczenie liczenia na palcach dla poznania matematycznego u człowieka dorosłego. *Psychologia–Etologia–Genetyka*, 30, 59–73.

Dąbrowski, M. (2006). *Pozwólmy dzieciom myśleć! O umiejętnościach matematycznych polskich trzecioklasistów*. Warszawa: Centralna Komisja Egzaminacyjna.

Filipiak, E. (ed.). (2008). *Rozwijanie zdolności uczenia się. Wybrane konteksty i problemy.* Bydgoszcz: Wydawnictwo Uniwersytetu Kazimierza Wielkiego.

Guidoni, P., Iannece, D. and Tortora, R. (2005). Multimodal language strategies activated by students in understanding and communicating mathematics. In M. Bosch (ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education, Sant Feliu de Guíxols* (pp. 841–851). Retrieved from http://www.mathematik.uni-dortmund.de/~erme/CERME4/CERME4_WG8.pdf

Ifrah, G. (1990). *Dzieje liczby, czyli historia wielkiego wynalazku*. Wrocław: Zakład Narodowy im. Ossolińskich.

Krygowska, Z. (1977). *Zarys dydaktyki matematyki*. Warszawa: Wydawnictwa Szkolne i Pedagogiczne.

Kvasz, L. (2014). Language in change: how we changed the language of mathematics and how the language of mathematics changed us. In M. Pytlak (ed.), *Communication in the mathematics classroom* (pp. 207–228). Rzeszów: Wydawnictwo Uniwersytetu Rzeszowskiego.

Morgan, C. (1998). *Writing mathematically, the discourse of investigation* (vol. 9: Studies in mathematics education series). London: Falmer Press.

Nowakowska, A., Orzechowska, M., Sosulska, D. and Zambrowska, M. (eds.). (2014). *Bydgoski bąbel matematyczny O wprowadzaniu zmian w nauczaniu matematyki w klasach I–III*. Warszawa: Instytut Badań Edukacyjnych.

Piaget, J. (1992). *Mowa i myślenie dziecka.* Warszawa: Wydawnictwo Naukowe PWN.

Pirie, S. (1998). Crossing the gulf between thought and symbol: language as (slippery) stepping-stones. In H. Steinbring, M. G. Bartolini Bussi and A. Sierpińska (eds.), *Language and communication in the mathematics classroom* (s. 7–29). Reston: National Council of Teachers of Mathematics.

Pirie, S. and Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics*, *9*(3), 7–11.

Radford, L. (2003). On culture and mind: a post-Vygotskian semiotic perspective. With an example from Greek mathematical thought. In M. Anderson, A. Saenz-Ludow, S. Zellweger and V. Cifarelli (eds.), *Perspective mathematics as semiotic: from thinking to interpreting to knowing* (pp. 49–79). Ottawa: Legas Publishing.

Reclik, R. (2012). Schematy graficzne w nauczaniu początkowym matematyki. *Auxilium Sociale Novum*, 3–4, 75–85.

Reclik, R. (2015). Tworzenie reprezentacji graficznych jako wstęp do formalnej matematyzacji. In H. Kąkol (ed.), *Współczesne problemy nauczania matematyki* (vol. 6: Prace monograficzne z dydaktyki matematyki, pp. 163–184). Bielsko-Biała: Forum Dydaktyków Matematyki.

Rożek, B. (2016). On formal and informal notation of calculation during the early learning of arithmetic by young students. *Annals of The Polish Mathematical Society, 5th Series: Didactica Mathematicae*, 38, 149–174.

Siwek, H. (1988). *Czynnościowe nauczanie matematyki*. Warszawa: Wydawnictwa Szkolne i Pedagogiczne.

Slezáková-Kratochvílová, J. and Swoboda, E. (2006). Kognitywne przeszkody w komunikowaniu się nauczyciel – uczeń. *Annals of The Polish Mathematical Society, 5th Series: Didactica Mathematicae*, 29, 185–207.

Stankiewicz, A. (2016). *Kompetencje studentów nauczania wczesnoszkolnego dotyczące umiejętności wizualizacji matematycznych problemów.* [Unpublished master thesis.] Jarosław: Państwowa Wyższa Szkoła Techniczno-Ekonomiczna.

Steinbring, H. (1997). Epistemological investigation of classroom interaction in elementary mathematics teaching. *Educational Studies in Mathematics*, *32*(1), 49–92.

Steinbring, H. (2005). The construction of new mathematical knowledge in classroom interaction. An epistemological perspective (series: Mathematics education library, vol. 38). Berlin–New York: Springer.

Struik, D. (1963). *Krótki zarys historii matematyki do końca XIX w.* Warszawa: Państwowe Wydawnictwo Naukowe.

Wygotski, L. S. (1989). *Myślenie i mowa* (series: Biblioteka klasyków psychologii). Warszawa: Państwowe Wydawnictwo Naukowe.

Acknowledgements

I would like to thank Jana Slezákova, Ph.D., from the Department of Mathematics and Mathematical Education of the Charles University in Prague (the Czech Republic) for the assistance in carrying out the observations. The problem described in this article was solved by pupils from one of Prague schools during classes conducted by students of the Pedagogical Department of the Charles University, constituting extracurricular activities provided under the supervision of doctor Slezákova.

A preliminary version of this article was published in Polish in *Edukacja*, *140*(1), 2017.