Podlasie 2014

Recent Results in Pure and Applied Mathematics

Edited by:
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Adam Grabowski
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Ewa Schmeidel

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Preface

The main purpose of this monograph is to present recent development in pure mathematics and its applications by the Podlasie mathematicians concentrated in two the most important scientific centres of the region (i.e. University of Białystok and Białystok Technical University), and by their colleagues from the other regions of Poland. These developments are related to algebra, geometry, and the equations on the different time scales (particularly, to the difference and differential equations). The editors hope that the monograph will stimulate the exchange of ideas both among authors of individual chapters and between the authors and readers of the monograph. They also hope that the monograph will stimulate further exploring of presented areas. Finally, they hope that the cooperation established during writing of this monograph; hopefully it will be continued in the future – possibly resulting in new achievements.

The volume consists of eleven chapters divided into three parts. Part I comprising algebraic results consists of five chapters.

Chapter 1 is devoted to the study of some subclasses of H-rings, i.e. rings in which every subring is an ideal. In the description of H-rings, the central role is played by the so-called almost null rings. The authors present and classify, up to an isomorphism, some general examples of non-torsion almost null rings.

Some class of homomorphically closed rings, called powerly hereditary rings, is studied in Chapter 2. It is proved that the Kurosh chain determined by this class stabilizes at step 3. With additional assumptions, the left Kurosh chain stabilizes at step 4. The facts presented are generalizations of the classical results for classes of nilpotent rings and for the hereditary classes.

New results for the structure of TI-groups are presented in Chapter 3. The structure theorem describing torsion TI-groups is proved, and the structure of the torsion part of mixed TI-groups is described as well. Furthermore, it is proved that every abelian torsion-free group of rank one is a TI-group. Numerous examples of TI-groups are given.

In Chapter 4, some known algebraic and geometric results concerning the spark of a matrix over an arbitrary field are recalled first. Then, a few new examples and observations are presented.

In Chapter 5, the author proves a particular case of the Hermite–Lindemann Transcendence Theorem saying that if α is a non-zero algebraic number, then e^{α} will be transcendental. Some applications of this theorem are also presented.

Part II is devoted to geometry and contains two chapters.

In Chapter 6 it is proved that a generalized Veronesian $V_k(M)$ cannot be realized in any Desarguesian projective space if $k \geq 3$ and the partial linear space M contains a line on at least 4 points or $k > 3$ and the partial linear space M contains a line on at least 3 points. This result is obtained using methods of the theory of combinatorial Veronesians. As a consequence of this fact, the author obtains that there is no Desarguesian projective space containing $V_k(P)$, where P is a projective space and $k \geq 3$. Also, the problem of realizability of $V_2(PG(2,2))$ in $PG(n,2)$ is solved.

The projective line $\mathbb{P}(R)$ over a finite associative ring R with unity is discussed in Chapter 7. It is defined as the set of free cyclic submodules of ${}^2R$, the two-dimensional left module over R. In particular, automorphisms and distant graphs of projective lines are discussed.

The last part (Part III) consisting of four chapters is focused on some topics in the field of difference and differential equations.

In Chapter 8, the author explorates the differences between asymptotic properties of solutions of difference equations and their continuous analogues on the example of the third order linear homogeneous differential and difference equations. It is shown that these equations (differential and recurrence) can have solutions with different properties concerning boundedness.

In Chapter 9, some three-dimensional nonlinear difference system with deviating arguments is studied, where the first equation of the system is a neutral type difference equation. The classification of nonoscillatory solutions of the considered system are presented. Also the sufficient conditions for boundedness of a nonoscillatory solution are given.

In Chapter 10, the symmetric and antisymmetric exponential functions in four variables are described, based on the same permutation group. Next, explicit formulas for the corresponding families of the orthogonal polynomials and some of their properties are derived like orthogonality, both the continuous one and the discrete one on a lattice. A general formula for the symmetric orthogonal polynomials is also given.

In Chapter 11, the problem of relative observability for a linear stationary fractional differential-algebraic delay system with jumps is investigated. Such a system consists of a fractional differential equation in the Caputo sense and an output equation. Using determining equation the author obtains effective parametric rank criteria for the relative observability. A dual controllability result is also formulated.

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Part I

Algebra

Recent Results in Pure and Applied Mathematics, Podlasie 2014
by A. Gomolińska, A. Grabowski, M. Hryniewicka, M. Kacprzak, E. Schmeidel (Eds.)

On the Non-torsion Almost Null Rings

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Abstract. The presented work is devoted to the study of some subclasses of H-rings, i.e. rings in which every subring is an ideal. In the description of H-rings a central role play the so-called almost null rings. In this paper we present and classify, up to isomorphism, some general examples of non-torsion almost null rings.

1 Introduction

All considered rings are associative, not necessarily with unity. A ring in which every subring is an ideal is called an H-ring. If the additive group R^+ of the ring R is a p-group, then we say that R is a p-ring and if furthermore R is a nil H-ring, we shall say that R is a nil-H-p-ring. The class of H-rings have been studied by a number of authors and the most important results were obtained by L. Rédei [7, 8], V. I. Andrijanov [1, 2] and R. Kruse [5, 6]. Thanks to their efforts, the problem of classification of H-rings has been reduced to the problem of classification of nil-H-p-rings for a prime integer p. To describe the class of nil-H-p-rings, mentioned authors used many types of rings defined by a complicated relations on generators. Unfortunately, the problem of classification of nil-H-p-rings (even rings from the same class), up to an isomorphism, is still open.

The most important subclass of the class of all nil-H-rings is the class of almost null rings, which was discovered by Kruse and independently by Andrijanov.

Definition 1 ([5], Definition 2.1). *We say that a ring R is* **almost null** *if for all $a, b \in R$ the following conditions are satisfied:*

(i) $a^3 = 0$,
(ii) $Ma^2 = 0$ *for some square-free integer M which depends on a,*
(iii) $ab = ka^2 = lb^2$ *for some integers k, l.*

The following important proposition is due to Kruse.

Proposition 1 ([5], Proposition 2.6). *A non-torsion nil-ring R is an H-ring if and only if R is an almost null ring.*

The class of almost null rings, is the first of listed, by Andrianov, classes of nil-H-p-rings (cf. [2], Definition 1), and occurs in the description of next presented by Andrianov classes. Moreover, almost null rings play a central role in the classification of the so-called filial rings (cf. [3, 4]).

2 Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{P}$ stand for the set of all positive integers, the set of all integers and the set of all primes, respectively. For $n \in \mathbb{N}$, let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the residue class ring modulo n.

In the current paper for a ring R we will use the following notation: for a subset S of R, we denote by $\langle S \rangle$, $[S]$, $a_R(S) = \{x \in R \ : \ xS = Sx = 0\}$ the subgroup of R^+ generated by S, the subring of R generated by S, the two-sided annihilator of S in R, respectively. Instead of $a_R(R)$ we will write $a(R)$, for short. Moreover, $\mathbb{T}(R) = \{x \in R \ : \ nx = 0$ for some $n \in \mathbb{N}\}$ and for any $p \in \mathbb{P}$: $R_p = \{x \in R : p^n x = 0$ for some $n \in \mathbb{N}\}$, $R[p] = \{x \in R \ : \ px^2 = 0\}$. For an abelian group M, by M^0 we denote the ring with zero multiplication and the additive group M and $M_p = \{a \in M \ : \ p^s a = 0$ for some $s \in \mathbb{N}\}$, for $p \in \mathbb{P}$.

Some characterizations of almost null rings where found by Kruse and Andrijanov, but are unsatisfactory due to lack of description up to an isomorphism. Now we present two theorems which are a reformulation and generalization of Kruse's theorem characterizing almost null rings (cf. [5], Proposition 2.10).

Theorem 1. *Let S be a ring and let p be a prime integer. Then S is an almost null ring such that $S = S[p]$ if and only if one of the following conditions is satisfied:*

(1) $S^2 = 0$,

(2) there exists $x \in S$ such that $x^2 \neq 0$, $px^2 = 0$, $px, x^2 \in a(S)$ and $S = \langle x \rangle + a(S)$,

(3) there exist $x, y \in S$ such that $S = \langle x, y \rangle + a(S)$, $x^2 \neq 0$, $px^2 = 0$, $px, py, x^2 \in a(S)$, $y^2 = Ax^2$, $xy = F_1 x^2$, $yx = F_2 x^2$, where $A, F_1, F_2 \in \mathbb{Z}$ and the congruence

$$X^2 + (F_1 + F_2)X + A \equiv 0 \pmod{p} \tag{1}$$

has no integer solution.

Moreover, if S is an almost null ring, then the quotient ring $S/a(S)$ is a $\mathbb{Z}_p$-algebra and $\dim_{\mathbb{Z}_p} S/a(S) \leq 2$.

Theorem 2. *A ring R is an almost null ring if and only if $R = \sum_{p \in \mathbb{P}} R[p]$, where for all distinct prime integers p, q we have $R[p] \cdot R[q] = 0$, $R[p] \lhd R$ and $R[p]$ satisfies one of the conditions (1), (2) or (3) of Theorem 1.*

Remark 1. Let p be an odd prime integer and let S_1 be a ring in which there exist elements x_1, y_1 such that $S_1 = \langle x_1, y_1 \rangle + a(S_1)$, $x_1^2 \neq 0$, $px_1^2 = 0$, $px_1, py_1, x_1^2 \in a(S_1)$, $y_1^2 = Ax_1^2$, $x_1 y_1 = -Fx_1^2$, $y_1 x_1 = Fx_1^2$ for some $A, F \in \mathbb{Z}$ such that $p \nmid A$, $p \nmid F$ and $-A$ is a quadratic non-residue modulo p. Let S_2 be a ring in which there exist elements X_1, Y_1 such that $S_2 = \langle X_1, Y_1 \rangle + a(S_2)$, $X_1^2 \neq 0$, $pX_1^2 = 0$, $pX_1, pY_1, X_1^2 \in a(S_2)$, $Y_1^2 = BX_1^2$, $X_1 Y_1 = -FX_1^2$, $Y_1 X_1 = FX_1^2$ for some $B \in \mathbb{Z}$ such that $p \nmid B$ and $-B$ is a quadratic non-residue modulo p. From Theorem 1 it follows that S_1, S_2 are almost null rings.

We will show that $S_1 \cong S_2$ implies that $A \equiv B \pmod p$. Assume that $f\colon S_1 \to S_2$ is a ring isomorphism. Then $f(x_1) = aX_1 + bY_1 + n_1$, $f(y_1) = cX_1 + dY_1 + n_2$ for some $n_1, n_2 \in a(S_2)$ and $a, b, c, d \in \mathbb{Z}$ such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc \not\equiv 0 \pmod p$ and:

$$\begin{cases} -F(aX_1 + bY_1 + n_1)^2 = (aX_1 + bY_1 + n_1)(cX_1 + dY_1 + n_2) \\ \ \ F(aX_1 + bY_1 + n_1)^2 = (cX_1 + dY_1 + n_2)(aX_1 + bY_1 + n_1) \\ \ \ A(aX_1 + bY_1 + n_1)^2 = (cX_1 + dY_1 + n_2)^2 \end{cases}$$

Hence

$$\begin{cases} -F(a^2 + Bb^2) \equiv ac - Fad + Fbc + bdB \pmod p \\ \ \ F(a^2 + Bb^2) \equiv ac - Fbc + Fad + bdB \pmod p \\ \ \ A(a^2 + Bb^2) \equiv c^2 + Bd^2 \pmod p \end{cases} \tag{2}$$

Adding the first two congruences and dividing by 2 we get $ac + bdB \equiv 0 \pmod p$. Since $p \nmid F$, so by the first congruence of (2) we obtain

$$a^2 + b^2B \equiv ad - bc \pmod p. \tag{3}$$

By the above and by the following identity

$$(a^2 + b^2B)(c^2 + d^2B) = (ac + bdB)^2 + B(ad - bc)^2$$

we get

$$(ad - bc)(c^2 + d^2B) \equiv B(ad - bc)^2 \pmod p.$$

But $p \nmid ad - bc$, so $c^2 + d^2B \equiv B(ad - bc) \pmod p$. Moreover, by the last congruence of (2), $A(a^2 + Bb^2) \equiv B(ad - bc) \pmod p$. From (3), $A(ad - bc) \equiv B(ad - bc) \pmod p$ and since $p \nmid ad - bc$, we get $A \equiv B \pmod p$.

Now, we present some technical results, which will be used in this paper.

Proposition 2. *Let x be any element of order p in a p-group $\langle a \rangle$. Then, for every abelian group B and for $A = \langle a \rangle \oplus B$ we have*

$$o(a) = max\{o(v) \ : \ v \in A_p \text{ and } x \in \langle v \rangle\}.$$

Proof. Assume, that there exists $v_0 \in A_p$ such that $x \in \langle v_0 \rangle$ and $o(v_0) = p^t > p^r = o(a)$. Since $o(x) = p$, so $x = Up^{t-1}v_0$ for some $U \in \mathbb{Z}$, $p \nmid U$. But $v_0 = ka + b$ for some $k \in \mathbb{Z}$ and $b \in B$. Therefore $p^{t-1}v_0 \in B$, so $x \in B$, a contradiction. Thus $o(a) = max\{o(v) : v \in A_p$ and $x \in \langle v \rangle\}$.

Theorem 3 (Walker). *Let F, F', G, H be abelian groups. If $F \oplus G \cong F' \oplus H$, $F \cong F'$ and F is finitely generated, then $G \cong H$.*

Proof. Corollary 8 of [9].

The next lemma follows directly from the Dirichlet's theorem on arithmetic progressions, but we give an elementary proof.

Lemma 1. *Let p be any prime integer and let a, b be any integers such that at least one of them is not divisible by p. Then there exist relatively prime integers x, y such that $x \equiv a \pmod{p}$ and $y \equiv b \pmod{p}$.*

Proof. Without loss of generality we may assume that $a, b \in \mathbb{Z}_p$ and $a \neq 0$. If $b = 0$ it is enough to set $x = a + p$ and $y = p$, because $p \nmid a$. Now, assume $b \neq 0$. Since the multiplicative group of the field $\mathbb{Z}_p$ is cyclic, there exists $c \in \mathbb{Z}_p \setminus \{0\}$ such that $a \equiv c^k \pmod{p}$ and $b \equiv c^l \pmod{p}$ for some $k, l \in \mathbb{N}$. Hence $p \nmid c$. Let $x = c^k + (c+1)p$ and $y = c^l$. Then $x \equiv a \pmod{p}$ and $y \equiv b \pmod{p}$. If the integers x and y were not relatively prime then for some prime we wolud have $q \mid c^l$ and $q \mid c^k + (c+1)p$. Thus $q \mid c$ and $q < p$. Therefore $q \mid (c+1)p$ and hence $q \mid c + 1$. But $q \mid c$, so we obtain $q \mid 1$, a contradiction.

Proposition 3. *Let p be any odd prime integer and let A, F, V_1, V_2 be integers such that $p \nmid F^2 - 4A$. Then*

$$\{X^2 + FXY + AY^2 + V_1X + V_2Y \ : \ X, Y \in \mathbb{Z}_p\} = \mathbb{Z}_p.$$

Proof. First, we will prove that $\{X^2 + FXY + AY^2 \ : \ X, Y \in \mathbb{Z}_p\} = \mathbb{Z}_p$. Since $p > 2$, so $1/2 \in \mathbb{Z}_p$ and for arbitrary $X, Y \in \mathbb{Z}_p$ we have

$$X^2 + FXY + AY^2 = (X + (F/2)Y)^2 - (F^2 - 4A)(Y/2)^2 .$$

Therefore it is enough to show that $\{U^2 - \Delta V^2 \ : \ U, V \in \mathbb{Z}_p\} = \mathbb{Z}_p$ for $\Delta = F^2 - 4A$. Take any $c \in \mathbb{Z}_p$ and consider the sets $A = \{U^2 - c \ : \ U \in \mathbb{Z}_p\}$ and $B = \{\Delta V^2 \ : \ V \in \mathbb{Z}_p\}$. Since the set $\{W^2 \ : \ W \in \mathbb{Z}_p\} = \{0^2, 1^2, \ldots, ((p-1)/2)^2\}$ has cardinality $(p+1)/2 > p/2$ and $p \nmid \Delta$, so both A and B have cardinality $(p+1)/2$. Hence A and B cannot be disjoint as subsets of $\mathbb{Z}_p$. Thus, there exist $U, V \in \mathbb{Z}_p$ such that $U^2 - c = \Delta V^2$. Therefore $c = U^2 - \Delta V^2$.

Now, we consider the general case, when the integers V_1 and V_2 are arbitrary. Since $p \nmid F^2 - 4A$, so there exist $a, b \in \mathbb{Z}$ such that $2a + Fb \equiv V_1 \pmod{p}$ and $Fa + 2Ab \equiv V_2 \pmod{p}$. Hence, for arbitrary $X, Y \in \mathbb{Z}$ we have $X^2 + FXY + AY^2 + V_1X + V_2Y \equiv (X+a)^2 + F(X+a)(Y+b) + A(Y+b)^2 - (a^2 + Fab + Ab^2) \pmod{p}$. Therefore, by the first part of the proof, we get $\{X^2 + FXY + AY^2 + V_1X + V_2Y \ : \ X, Y \in \mathbb{Z}_p\} = \mathbb{Z}_p$.

3 The main examples

Example 1. Let p be any prime integer and let integers F_1, F_2, A be such that the congruence (1) has no solution. Let $U \in \mathbb{Z}_p \setminus \{0\}$ and $m, n \in \mathbb{N}$, wherein $n > 1$. Moreover, let $R = \mathbb{Z}^+_{p^m} \times \mathbb{Z}^+_{p^n}$ or $R = \mathbb{Z}^+ \times \mathbb{Z}^+_{p^n}$. In the group R we define a multiplication by the formula

$$(k_1, l_1) \cdot (k_2, l_2) = (0, U \cdot (k_1l_2F_2 + l_1k_2F_1 + Ak_1k_2 + l_1l_2) \cdot p^{n-1}). \qquad (4)$$

A standard computation shows that this multiplication is well-defined, distributive over addition and $(ab)c = a(bc) = 0$ for any $a, b, c \in R$. The ring constructed above will be dentoted by

$$(\mathbb{Z}_{p^m} \times_{Up^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A} \quad \text{or} \quad (\mathbb{Z} \times_{Up^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A}.$$

Since, the congruence (1) has no solution, so $\langle (k,l)^2\rangle = \langle (0,p^{n-1})\rangle$ if $p \nmid k$ or $p \nmid l$ and $(k,l)^2 = (0,0)$ if $p \mid k$ and $p \mid l$.

Let $y = (1,0)$, $x = (0,1)$. Then $R^+ = \langle y\rangle \oplus \langle x\rangle$, $x^2 = U \cdot p^{n-1}x$, $y^2 = Ax^2$, $xy = F_1x^2$, $yx = F_2x^2$, $o(x^2) = p$ and $x^2, px, py \in a(R)$. By Theorem 1, R is an almost null ring. Moreover $a(R) = \langle py\rangle \oplus \langle px\rangle$, $(R/a(R))^+ \cong \mathbb{Z}_p^+ \times \mathbb{Z}_p^+$.

Note that there exists $V \in \mathbb{Z}$ such that $UV \equiv 1 \pmod p$. Let $F_1' = UF_1$, $F_2' = UF_2$, $A' = U^2A$. Then the congruence $X^2 + (F_1'+F_2')X + A' \equiv 0 \pmod p$, has no solution. Moreover, the functions

$$f\colon (\mathbb{Z}_{p^m} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1',F_2',A'} \to (\mathbb{Z}_{p^m} \times_{Up^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A},$$

$$g\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1',F_2',A'} \to (\mathbb{Z} \times_{Up^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A}$$

given by the formulas $f((k,l)) = (k, Vl)$, $g((k,l)) = (k, Vl)$ are ring isomorphisms. Therefore, without loss of generality, we may assume that $U = 1$.

Furthermore, if an integer W is not divisible by p, then there exists $V \in \mathbb{Z}$ such that $WV \equiv 1 \pmod p$ and the congruence $X^2 + (WF_1 + WF_2)X + W^2A \equiv 0 \pmod p$ has no solution. Moreover, the function $f\colon (\mathbb{Z}_{p^m} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A} \to (\mathbb{Z}_{p^m} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{WF_1,WF_2,W^2A}$ given by the formula $f((k,l)) = (Vk, l)$ is a ring isomorphism.

Example 2. Let p, U, F_1, F_2, A be the same as in the Example 1. Let $m, n, s \in \mathbb{N}$. Moreover, let $R = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^s}$ or $R = \mathbb{Z} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^s}$ or $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{p^s}$. In the group R we define a multiplication by the formula

$$(k_1,l_1,t_1) \cdot (k_2,l_2,t_2) = (0,0,U(k_1l_2F_2 + l_1k_2F_1 + Ak_1k_2 + l_1l_2) \cdot p^{s-1}) \qquad (5)$$

A standard computation shows that this multiplication is well-defined, distributive over addition and $(ab)c = a(bc) = 0$ for any $a, b, c \in R$. The ring constructed above will be denoted, respectively, by

$$(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{Up^{s-1}} \mathbb{Z}_{p^s},$$

$$(\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{Up^{s-1}} \mathbb{Z}_{p^s},$$

$$(\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{Up^{s-1}} \mathbb{Z}_{p^s}.$$

Since, the congruence (1) has no solution, so $(k,l,t)^2 = (0,0,0)$ if and only if $p \mid k$ and $p \mid l$. Moreover, $\langle (k,l,t)^2\rangle = \langle (0,0,p^{s-1})\rangle$ if $p \nmid k$ or $p \nmid l$.

Let $y = (1,0,0)$, $x = (0,1,0)$, $z = (0,0,1)$. Then $x^2 = Up^{s-1}z$, $y^2 = Ax^2$, $xy = F_1x^2$, $yx = F_2x^2$, $a(R) = \langle py\rangle \oplus \langle px\rangle \oplus \langle z\rangle$, $R = \langle x, y\rangle + a(R)$, $(R/a(R))^+ \cong \mathbb{Z}_p^+ \times \mathbb{Z}_p^+$, so by Theorem 1 it follows that R is an almost null ring.

Moreover, the functions

$$f\colon (\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{Up^{s-1}} \mathbb{Z}_{p^s},$$

$$g\colon (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{Up^{s-1}} \mathbb{Z}_{p^s},$$

$$h\colon (\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{Up^{s-1}} \mathbb{Z}_{p^s},$$

given by the formulas $f((k,l,t)) = (k,l,Ut)$, $g((k,l,t)) = (k,l,Ut)$, $h((k,l,t)) = (k,l,Ut)$ are ring isomorphisms. Therefore, without loss of generality, we may assume that $U = 1$.

Furthermore, if an integer W is not divisible by p, then the congruence $X^2 + (WF_1 + WF_2)X + W^2A \equiv 0 \pmod p$ has no solution and the functions

$$f\colon (\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})_{WF_1,WF_2,W^2A} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

$$g\colon (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z}_{p^n})_{WF_1,WF_2,W^2A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$$

given by the formulas $f((k,l,t)) = (k,Wl,W^2t)$, $g((k,l,t)) = (k,Wl,W^2t)$ are ring isomorphisms.

4 The rings of the form $(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A}$ and their extensions

Proposition 4. *For any odd prime integer p and for $i = 1,2$ let $R_i = (\mathbb{Z} \times_{p^{n_i-1}} \mathbb{Z}_{p^{n_i}})_{0,0,-\mu_i}$, where $n_i > 1$, μ_i is a fixed quadratic non-residue modulo p. Let A_i be a torsion ring with zero multiplication. Then $R_1 \oplus A_1 \cong R_2 \oplus A_2$ if and only if $n_1 = n_2$, $A_1^+ \cong A_2^+$ and $\mu_1 \equiv \mu_2 \pmod p$.*

Proof. Implication $\Leftarrow$ is obvious. For the implication $\Rightarrow$, assume that $f\colon R_1 \oplus A_1 \to R_2 \oplus A_2$ is a ring isomorphism. By Proposition 2, $p^{n_i} = max\{o(v) : (R_i \oplus A_i)^2 \subseteq \langle v\rangle\}$ for $i = 1,2$, hence $n_1 = n_2 = n$. Moreover, f is an isomorphism of the additive groups, so by Theorem 3, $A_1^+ \cong A_2^+$. Next, $o(f((0,1))) = o((0,1)) = p^n$, so there exist $l,d \in \mathbb{Z}$ and $\alpha \in A_2$ such that $f((0,1)) = (l, d\cdot 1) + \alpha$. But $(0,1) \notin a(R_1 \oplus A_1)$ and $o((0,1)) < \infty$. Thus $l = 0$ and $(0, d\cdot 1) + \alpha \notin a(R_2 \oplus A_2)$. Hence $p \nmid d$. Moreover $f((1,0)) = (a, b\cdot 1) + \beta$ for some $a,b \in \mathbb{Z}$, $\beta \in A_2$. Therefore $f((k,l) + \gamma) = (ka, (kb + ld)\cdot 1) + l\alpha + k\beta + f(\gamma)$ for arbitrary $k \in \mathbb{Z}$, $l \in \mathbb{Z}_{p^n}$, $\gamma \in A_1$. But $l\alpha + k\beta + f(\gamma) \in A_2 \oplus \mathbb{T}(R_2)$, the group A_1^+ is torsion and f is "onto", so $ak = 1$ for some $k \in \mathbb{Z}$. Hence $a = \pm 1$ and $a^2 = 1$.

Next, $(0,1)^2 = (0,p^{n-1})$, so $f((0,1)^2) = p^{n-1}\cdot f((0,1)) = p^{n-1}\cdot(0,d\cdot 1) + p^{n-1}\alpha$ and $f((0,1)^2) = [f((0,1))]^2 = ((0,d\cdot 1)+\alpha)^2 = (0, d^2\cdot p^{n-1})$. Therefore $d^2 \equiv d \pmod p$ and since $p \nmid d$, so $d \equiv 1 \pmod p$.

Moreover, $(0,0) = f((0,0)) = f((1,0)\cdot(0,1)) = f((1,0))\cdot f((0,1)) = ((a, b\cdot 1) + \beta)\cdot((0,d\cdot 1)+\alpha) = (0, bd\cdot p^{n-1})$, so $bd \equiv 0 \pmod p$. By the above $p \mid b$. Applying the function f to $(1,0)^2 = -\mu_1(0,1)$ we get $-\mu_1 d \equiv -\mu_2 a^2 + b^2 \pmod p$. But $a^2 = 1$, $d \equiv 1 \pmod p$ and $p \mid b$, so $\mu_1 \equiv \mu_2 \pmod p$.

Proposition 5. *Let p be an odd prime integer and μ_1, μ_2 be a fixed quadratic non-residues modulo p. Then for every integer $n > 1$,*

$$(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,-\mu_1} \oplus \mathbb{Z}^0 \cong (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,-\mu_2} \oplus \mathbb{Z}^0.$$

Proof. Elementary number theory implies that $\mu_1 \equiv a^2\mu_2 \pmod p$ for some $a \in \mathbb{Z}$ and that there exist $C,E \in \mathbb{Z}$ such that $aC - pE = 1$. It is easy to check that the function

$$f\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,-\mu_1} \oplus \mathbb{Z}^0 \to (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,-\mu_2} \oplus \mathbb{Z}^0,$$

given by the formula $f(k,l,t) = (ak+pt, l, Ct+Ek)$ is a ring isomorphism.

Proposition 6. *Let $n>1$ be an integer. Then*

$$(\mathbb{Z} \times_{2^{n-1}} \mathbb{Z}_{2^n})_{0,1,1} \cong (\mathbb{Z} \times_{2^{n-1}} \mathbb{Z}_{2^n})_{1,0,1}.$$

Proof. It is easy to check that the function

$$f\colon (\mathbb{Z}_{2^m} \times_{2^{n-1}} \mathbb{Z}_{2^n})_{1,0,1} \to (\mathbb{Z}_{2^m} \times_{2^{n-1}} \mathbb{Z}_{2^n})_{0,1,1},$$

given by the formula $f((k,l)) = (k, (k+l)\cdot 1)$ is a ring isomorphism.

Proposition 7. *Let p be an odd prime integer and $n>1$ be an integer. Let F, A be integers such that $p \nmid F$ and the congruence $X^2 + 2FX + A \equiv 0 \pmod{p}$ has no solution. Then*

$$(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,A-F^2} \cong (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F,F,A}.$$

Proof. It is easy to check that the function

$$f\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,A-F^2} \to (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F,F,A},$$

given by the formula $f((k,l)) = (k, (l-Fk)\cdot 1)$ is a ring isomorphism.

Proposition 8. *Let p be an odd prime integer and $n>1$ be an integer. Let A, F_1, F_2 be integers such that $p \nmid F_1 - F_2$ and the congruence $X^2 + (F_1 + F_2)X + A \equiv 0 \pmod{p}$ has no solution. Then there exist integers G, B such that $4B \equiv 4A - (F_1+F_2)^2 \pmod{p}$, $2G \equiv F_2 - F_1 \pmod{p}$ and*

$$(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A} \cong (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-G,G,B}$$

Proof. Since p is odd, so there exist $b, G, B \in \mathbb{Z}$ such that $2b \equiv -(F_1+F_2) \pmod{p}$, $4B \equiv 4A - (F_1+F_2)^2 \pmod{p}$ and $2G \equiv F_2 - F_1 \pmod{p}$. It is easy to check that the function

$$f\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-G,G,B} \to (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A},$$

given by the formula $f((k,l)) = (k, (kb+l)\cdot 1)$ is a ring isomorphism.

Proposition 9. *Let p be an odd prime integer and $n>1$ be an integer. Let μ be a fixed quadratic non-residue modulo p. Let G and F be integers, which both are not divisible by p. The rings $(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-G,G,-\mu}$ and $(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-F,F,-\mu}$ are isomorphic if and only if $G \equiv F \pmod{p}$ or $G \equiv -F \pmod{p}$.*

Proof. Assume that $G \equiv F \pmod{p}$ or $G \equiv -F \pmod{p}$. Then $F \equiv UG \pmod{p}$ for $U = \pm 1$. It is easy to check that the function

$$f\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-F,F,-\mu} \to (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-G,G,-\mu},$$

given by the formula $f((k,l)) = (Uk, l)$ is a ring isomorphism.

Conversely, let $g\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-G,G,-\mu} \to (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-F,F,-\mu}$ be a ring isomorphism. Then $o(g((0,1))) = o((0,1)) = p^n$, so $g((0,1)) = (0,c)$ for some $c \in \mathbb{Z}_{p^n}$. But $(0,1)^2 = p^{n-1}(0,1)$, so $(0,c)^2 = p^{n-1}(0,c)$. Moreover $(0,c)^2 = c^2(0,1)^2 = c^2 p^{n-1}(0,1)$. Hence $c^2 \equiv c \pmod{p}$. Since $f((0,1)^2) \neq (0,0)$, so $p \nmid c$ and $c \equiv 1 \pmod{p}$. Next, $g((1,0)) = (a,b)$ for some $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{p^n}$. Therefore $(1,0) = K(a,b) + L(0,c)$ for some $K, L \in \mathbb{Z}$ and hence $a = \pm 1$. Moreover $(a,b)^2 = -\mu p^{n-1}(0,1)$ and $(a,b)^2 = (0, (-\mu a^2 + b^2)p^{n-1}) = (0, (-\mu + b^2)p^{n-1})$, so $-\mu + b^2 \equiv -\mu \pmod{p}$ and consequently $p \mid b$. Next, $g((1,0) \cdot (0,1)) = g((1,0)) \cdot g((0,1))$, so $g(Gp^{n-1} \cdot (0,1)) = (a,b) \cdot (0,c)$. Thus $Gc \equiv acF + bc \pmod{p}$. But $p \mid b$ and $p \nmid c$, so $G \equiv aF \pmod{p}$. Finally $G \equiv \pm F \pmod{p}$.

Proposition 10. *Let p be an odd prime integer and let μ be a fixed quadratic non-residue modulo p. Let $n > 1$ be an integer. Then for every integer F which is not divisible by p,*

$$(\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-1,1,-\mu} \oplus \mathbb{Z}^0 \cong (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-F,F,-\mu F^2} \oplus \mathbb{Z}^0.$$

Proof. There exist $A, M \in \mathbb{Z}$ such that $FA + Mp = 1$. It is easy to check, that the function

$$f\colon (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-1,1,-\mu} \oplus \mathbb{Z}^0 \to (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-F,F,-\mu F^2} \oplus \mathbb{Z}^0,$$

given by the formula $f((k,l,t)) = (kA + pt, l, Ft - Mk)$ is a ring isomorphism.

By Example 1, Propositions 4, 6, 7, 8, 9 and Remark 1 we get the following theorem.

Theorem 4. *Let p, n, A, F_1, F_2 be the same as in the Example 1. If $p > 2$, then let μ be a fixed quadratic non-residue modulo p. Let $R = (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{F_1,F_2,A}$. Then:*

(i) if $p > 2$ and the ring R is commutative, then $F_1 \equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{0,0,-\mu V^2},$$

for exactly one $V \in \{1, 2, \ldots, (p-1)/2\}$,

(ii) if $p = 2$, then the ring R is not commutative and

$$R \cong (\mathbb{Z} \times_{2^{n-1}} \mathbb{Z}_{2^n})_{0,1,1},$$

(iii) if $p > 2$ and the ring R is not commutative, then $F_1 \not\equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times_{p^{n-1}} \mathbb{Z}_{p^n})_{-F,F,-\mu V^2},$$

for exactly one pair $(F, V) \in \{1, 2, \ldots, (p-1)/2\} \times \{1, 2, \ldots, (p-1)/2\}$.

5 The rings of the form $(\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$

Proposition 11. *Let p, n, s, A, F_1, F_2 be the same as in the Example 2. Let $R = (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$. Then:*

(i) if $p > 2$ and the ring R is commutative, then $F_1 \equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z}_{p^n})_{0,0,A-F_1^2} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

(ii) if $p = 2$, then the ring R is not commutative and

$$R \cong (\mathbb{Z} \times \mathbb{Z}_{2^n})_{0,1,1} \times_{2^{s-1}} \mathbb{Z}_{2^s} \cong (\mathbb{Z} \times \mathbb{Z})_{1,0,1} \times_{2^{s-1}} \mathbb{Z}_{2^s},$$

(iii) if $p > 2$ and the ring R is not commutative, then $F_1 \not\equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z}_{p^n})_{-G,G,B} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

for all $G, B \in \mathbb{Z}$ such that $4B \equiv 4A - (F_1 + F_2)^2 \pmod{p}$, $2G \equiv F_2 - F_1 \pmod{p}$.

Proof. (i). It is enough to check, that the function

$$F \colon (\mathbb{Z} \times \mathbb{Z}_{p^n})_{0,0,A-F_1^2} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_1,A} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

given by the formula $F((k, l, t)) = (k, l - F_1k, t)$, is a ring isomorphism.

(ii). The function

$$F \colon (\mathbb{Z} \times \mathbb{Z}_{2^n})_{1,0,1} \times_{2^{s-1}} \mathbb{Z}_{2^s} \to (\mathbb{Z} \times \mathbb{Z}_{2^n})_{0,1,1} \times_{2^{s-1}} \mathbb{Z}_{2^s},$$

given by the formula $F((k, l, t)) = (k, k + l, t)$ is a ring isomorphism.

(iii). Since p is an odd prime, so there exist $b, G, B \in \mathbb{Z}$ such that $2b \equiv -(F_1 + F_2) \pmod{p}$, $4B \equiv 4A - (F_1 + F_2)^2 \pmod{p}$ and $2G \equiv F_2 - F_1 \pmod{p}$. It is easy to check that the function

$$f \colon (\mathbb{Z} \times \mathbb{Z}_{p^n})_{-G,G,B} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$$

given by the formula $f((k, l, t)) = (k, kb + l, t)$ is a ring isomorphism.

By Remark 1, Example 2 and Proposition 11 we get the following theorem.

Theorem 5. *Let p, n, s, A, F_1, F_2 be the same as in the Example 2. If $p > 2$, then let μ be a fixed quadratic non-residue modulo p. Let $R = (\mathbb{Z} \times \mathbb{Z}_{p^n})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$. Then:*

(i) if $p > 2$ and the ring R is commutative, then $F_1 \equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z}_{p^n})_{0,0,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

(ii) if $p = 2$, then the ring R is not commutative and

$$R \cong (\mathbb{Z} \times \mathbb{Z}_{2^n})_{0,1,1} \times_{2^{s-1}} \mathbb{Z}_{2^s},$$

(iii) if $p > 2$ and the ring R is not commutative, then $F_1 \not\equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z}_{p^n})_{-1,1,-\mu V^2} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

for exactly one $V \in \{1, 2, \ldots, (p-1)/2\}$.

6 The rings of the form $(\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$

Proposition 12. *Let p, s, A, F_1, F_2 be the same as in the Example 2 and let $R = (\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$. Then:*

(i) if $p > 2$ and the ring R is commutative, then $F_1 \equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z})_{0,0,A-F_1^2} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

(ii) if $p = 2$, then the ring R is not commutative and

$$R \cong (\mathbb{Z} \times \mathbb{Z})_{0,1,1} \times_{2^{s-1}} \mathbb{Z}_{2^s} \cong (\mathbb{Z} \times \mathbb{Z})_{1,0,1} \times_{2^{s-1}} \mathbb{Z}_{2^s},$$

(iii) if $p > 2$ and the ring R is not commutative, then $F_1 \not\equiv F_2 \pmod{p}$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z})_{-G,G,B} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

for all $G, B \in \mathbb{Z}$ such that $4B \equiv 4A - (F_1 + F_2)^2 \pmod{p}$, $2G \equiv F_2 - F_1 \pmod{p}$.

Proof. (i). It is easy to check that the function

$$F\colon (\mathbb{Z} \times \mathbb{Z})_{0,0,A-F_1^2} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z})_{F_1,F_1,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$$

given by the formula $F((k, l, t)) = (k, l - F_1 k, t)$ is a ring isomorphism.

(ii). The function

$$F\colon (\mathbb{Z} \times \mathbb{Z})_{1,0,1} \times_{2^{s-1}} \mathbb{Z}_{2^s} \to (\mathbb{Z} \times \mathbb{Z})_{0,1,1} \times_{2^{s-1}} \mathbb{Z}_{2^s}$$

given by the formula $F((k, l, t)) = (k, k + l, t)$ is a ring isomorphism.

(iii). Since p is an odd prime, so there exist $b, G, B \in \mathbb{Z}$ such that $2b \equiv -(F_1+F_2) \pmod{p}$, $4B \equiv 4A-(F_1+F_2)^2 \pmod{p}$ and $2G \equiv F_2 - F_1 \pmod{p}$. It Is easy to check, that the function

$$f\colon (\mathbb{Z} \times \mathbb{Z})_{-G,G,B} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$$

given by the formula $f((k, l, t)) = (k, kb + l, t)$ is a ring isomorphism.

Proposition 13. *Let p be an odd prime integer and μ be a fixed quadratic non-residue modulo p. Let s be a natural number and t an integer, which is not divisible by p. Then*

$$(\mathbb{Z} \times \mathbb{Z})_{0,0,-\mu t^2} \times_{p^{s-1}} \mathbb{Z}_{p^s} \cong (\mathbb{Z} \times \mathbb{Z})_{0,0,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s}.$$

Proof. By Lemma 1 and Proposition 3 there exist relatively prime integers k_1 and k_2 such that

$$k_1^2 - \mu t^2 k_2^2 \equiv t \pmod{p}. \tag{6}$$

Moreover, there exists $s \in \mathbb{Z}$ such that $st \equiv 1 \pmod{p}$. Hence from (6) we get

$$sk_1^2 - \mu t k_2^2 \equiv 1 \pmod{p}. \tag{7}$$

Hence, there exists $W_0 \in \mathbb{Z}$ such that

$$sk_1^2 - \mu t k_2^2 = 1 + pW_0. \tag{8}$$

Since the integers k_1 and k_2 are relatively prime, so there exist $U, V \in \mathbb{Z}$ such that $k_1 U - k_2 V = -W_0$. Let

$$l_1 = \mu t k_2 + pV, \quad l_2 = sk_1 + pU. \tag{9}$$

By formulas (8) and (9) we obtain

$$k_1 l_2 - l_1 k_2 = 1. \tag{10}$$

But $st \equiv 1 \pmod{p}$, so by the formula (9) we get

$$\mu k_1 k_2 \equiv l_1 l_2 \pmod{p}. \tag{11}$$

Next, by (9) and the fact that $st \equiv 1 \pmod{p}$ we get $-\mu t^2(l_2^2 - \mu k_2^2) \equiv -\mu k_1^2 + \mu^2 t^2 k_2^2 \equiv -\mu(k_1^2 - \mu t^2 k_2^2) \pmod{p}$ and $l_1^2 - \mu k_1^2 \equiv \mu^2 t^2 k_2^2 - \mu k_1^2 \equiv -\mu(k_1^2 - \mu t^2 k_2^2) \pmod{p}$. Therefore

$$l_1^2 - \mu k_1^2 \equiv -\mu t^2 (l_2^2 - \mu k_2^2) \pmod{p}. \tag{12}$$

By (10) it follows that the function $g\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ given by the formula $g((K, L)) = (Kk_2 + Lk_1, Kl_2 + Ll_1)$ is a group isomorphism. Combining this fact and formulas (10)–(12) one can show that the function

$$f\colon (\mathbb{Z} \times \mathbb{Z})_{0,0,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z})_{0,0,-\mu t^2} \times_{p^{s-1}} \mathbb{Z}_{p^s}$$

given by $f((K, L, T)) = (Kk_2 + Lk_1, Kl_2 + Ll_1, t \cdot T)$ is a ring isomorphism.

Proposition 14. *Let p be an odd prime integer and μ a fixed quadratic non-residue modulo p. Let s be a natural number and F an integer, which is not divisible by p. Then*

$$(\mathbb{Z} \times \mathbb{Z})_{-1,1,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s} \cong (\mathbb{Z} \times \mathbb{Z})_{-F,F,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s}.$$

Proof. By Lemma 1 and Proposition 3 there exist relatively prime integers k_1 and l_1 such that

$$l_1^2 - \mu k_1^2 \equiv F \pmod{p}. \tag{13}$$

Since $p \nmid F$, so $sF \equiv 1 \pmod{p}$ for some $s \in \mathbb{Z}$. Hence from (13) we get

$$sl_1^2 - s\mu k_1^2 \equiv 1 \pmod{p}. \tag{14}$$

Thus, there exists $W_0 \in \mathbb{Z}$ such that $sl_1^2 - s\mu k_1^2 = 1 + pW_0$. But the integers k_1 and l_1 are relatively prime, so there exist $U, V \in \mathbb{Z}$ such that $k_1 V - l_1 U = W_0$. Let

$$k_2 = sl_1 + pU, \quad l_2 = s\mu k_1 + pV. \tag{15}$$

Thus $k_1 l_2 - k_2 l_1 = k_1(s\mu k_1 + pV) - (sl_1 + pU)l_1 = -(sl_1^2 - s\mu k_1^2) + p(k_1 V - l_1 U) = -1 - pW_0 + pW_0 = -1$, so

$$k_1 l_2 - k_2 l_1 = -1. \tag{16}$$

The formula (15) implies $\mu k_1 k_2 \equiv s\mu k_1 l_1 \pmod p$ and $l_1 l_2 \equiv s\mu k_1 l_1 \pmod p$. Therefore

$$\mu k_1 k_2 \equiv l_1 l_2 \pmod p. \tag{17}$$

But $sF \equiv 1 \pmod p$, so by formulas (13) and (15) we have $-\mu(l_1^2 - \mu k_1^2) \equiv -\mu F \pmod p$ and $l_2^2 - \mu k_2^2 \equiv s^2\mu^2 k_1^2 - \mu s^2 l_1^2 \equiv -\mu s^2(l_1^2 - \mu k_1^2) \equiv -\mu s^2 F \equiv -\mu F \pmod p$, so

$$l_2^2 - \mu k_2^2 \equiv -\mu(l_1^2 - \mu k_1^2) \pmod p. \tag{18}$$

By (16) it follows that the function $g\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ given by the formula $g((K, L)) = (Kk_2 + Lk_1, Kl_2 + Ll_1)$ is a group isomorphism. By the above and by formulas (16)–(18) one can show that the function

$$f\colon (\mathbb{Z} \times \mathbb{Z})_{-1,1,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s} \to (\mathbb{Z} \times \mathbb{Z})_{-F,F,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s}$$

given by the formula $f((K, L, T)) = (Kk_2 + Lk_1, Kl_2 + Ll_1, F \cdot T)$ is a ring isomorphism.

From Propositions 12, 13, 14 and Remark 1 we get the following theorem.

Theorem 6. *Let p, s, A, F_1, F_2 be the same as in the Example 2. If $p > 2$, then let μ be a fixed quadratic non-residue modulo p. Let $R = (\mathbb{Z} \times \mathbb{Z})_{F_1,F_2,A} \times_{p^{s-1}} \mathbb{Z}_{p^s}$. Then:*

(i) if $p > 2$ and the ring R is commutative, then $F_1 \equiv F_2 \pmod p$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z})_{0,0,-\mu} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

(ii) if $p = 2$, then the ring R is not commutative and

$$R \cong (\mathbb{Z} \times \mathbb{Z})_{0,1,1} \times_{2^{s-1}} \mathbb{Z}_{2^s},$$

(iii) if $p > 2$ and the ring R is not commutative, then $F_1 \not\equiv F_2 \pmod p$ and

$$R \cong (\mathbb{Z} \times \mathbb{Z})_{-1,1,-\mu V^2} \times_{p^{s-1}} \mathbb{Z}_{p^s},$$

for exactly one $V \in \{1, 2, \ldots, (p-1)/2\}$.

References

1. Andrijanov, V. I.: Mixed Hamiltonian nilrings. (Russian) Ural. Gos. Univ. Mat. Zap. 5(3), 15–30 (1966)
2. Andrijanov, V. I.: Periodic Hamiltonian rings. Mat. Sb. (N.S.) 74(116), 241–261 (1967); translation in Mat. Sbornik 74(116) No. 2, 225–241 (1967)

3. Andruszkiewicz, R. R., Pryszczepko, K.: The classification of commutative noetherian, filial rings with identity. Comm. Algebra. 40, 1690–1703 (2012)
4. Andruszkiewicz, R. R., Pryszczepko, K.: The classification of commutative torsion filial rings. J. Aust. Math. Soc. 95, no. 3, 289–296 (2013)
5. Kruse, R. L.: Rings in which all subrings are ideals. I. Canad. J. Math., 20, 862–871 (1968).
6. Kruse, R. L.: Rings with periodic additive group in which all subrings are ideals. Dissertation, California Institute of Technology, (1964).
7. Rédei, L.: Vollidealringe im weiteren Sinn. I. Acta Math. Acad. Sci. Hungar. 3, 243–268 (1952).
8. Rédei, L.: Die Vollidealringe. Monatsh. Math. 56, 89–95 (1952).
9. Walker, E. A.: Cancellation in direct sums of groups. Proc. Amer. Math. Soc. 7, 898–902 (1956).

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The Powerly Hereditary Property in the Lower Radical Construction

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Abstract. A homomorphically closed class $\mathcal{M}$ of rings is called powerly hereditary if for every ring R in the class $\mathcal{M}$ there is an integer n greater than 1 such that $R^n \in \mathcal{M}$. In this paper we prove that Kurosh's chain determined by that class stabilises at step 3. With additional assumptions, left Kurosh's chain stabilises at step 4. Presented facts are a generalization of the classical results for classes of nilpotent rings and for hereditary classes.

Keywords: stabilisation of Kurosh's chains, powerly hereditary class, strong radical

All rings in this paper are associative but are not required to have a unity or to be commutative.

This paper deals with the stabilisation of Kurosh's chains within some of the classes of rings. This concept appeared in a radical theory and it is connected with description of lower radicals. For details of radical theory the readers are referred to [9].

In 1966 R. Anderson, N. Divinsky and A. Suliński found characterisations of the classes of Kurosh's chains using accessible subrings and showed the important usage of this concept (see [16]). On the basis of these characterisations they showed that Kurosh's chain stabilises on the first infinite ordinal number ω_0. In 1968 A. Heinicke in [5] gave an example of a class for which Kurosh's chain does not stabilise at any finite step and therefore he fullfiled the results presented in [16]. In [16] R. Anderson, N. Divinsky and A. Suliński asked the question if for any positive integer n there is a class for which Kurosh's chain stabilises at the step n. The problem appeared to be difficult and was completly solved in 1982 by K. I. Beidar in [6]. In [3] the autors constructed new examples of classes for which Kurosh's chain stabilises at exactly fixed step.

The most valuable results concerning the stabilisation of left Kurosh's chains were obtained by Divinsky, Krempa and Suliński in [7] (they studied onesided Kurosh's chains, which are some generalization of left Kurosh's chains). In [7] it was shown, among others, that if $\mathcal{M}$ is homomorphically closed, hereditary and contains all zero rings, then left Kurosh's chain determined by the class $\mathcal{M}$ stabilises at ω_0 (Theorem 6). In 1987 Puczyłowski [13] reinforced some of these results, showing that if $\mathcal{M}$ is a homorphically closed class of rings, then $\mathcal{M}^\alpha$

is a radical class for every $\alpha \geq \omega_0$. Puczyłowski showed that if $\mathcal{M}$ is a radical class, then $\mathcal{M}^\alpha$ is a radical class for every α and if $\mathcal{M}$ is also hereditary, then left Kurosh's chain stabilises at step 2. In 1990, by modifying Beidar's example, Andruszkiewicz and Puczyłowski [2] found for each ordinal $\gamma \leq \omega_0$ a homomorphically closed class $\mathcal{M}$ of rings, for which left Kurosh's chain stabilises at exactly γ-step. In 2012 we have constructed a radical class for which left Kurosh's chain does not stabilise at any finite step (Theorem 3.6, [4]).

If R is a ring then $A <_l R$ ($A \lhd R$) will mean that A is a left (two-sided) ideal of R. If A is a subring of R we use A_R to denote the ideal of R generated by A. Given a class $\mathcal{M}$ of rings, we denote by $l(\mathcal{M})$ the lower radical determined by the class $\mathcal{M}$.

Definition 1. *A radical $\mathcal{S}$ is said to be left strong if for every $A <_l R$ if $A \in \mathcal{S}$ then $A_R \in \mathcal{S}$.*

Important examples of left strong radicals are the Baer radical and the Jacobson radical. On the other hand the question if the nil-radical is left strong is the famous still open Koethe's problem.

Given a nonempty homomorphically closed class $\mathcal{M}$ of rings, we denote by $ls(\mathcal{M})$ the smallest left strong radical containing $\mathcal{M}$. We call it the *lower left strong radical determined by* $\mathcal{M}$.

Definition 2. *Let n be a positive integer. A subring A of a ring R is said to be n-accessible (left n-accessible) in R if there are subrings $R = A_0, A_1, \dots, A_{n-1}, A_n = A$ of R such that $A_i \lhd A_{i-1}$ ($A_i <_l A_{i-1}$) for $i = 1, 2, \dots, n$. A subring A is said to be accessible (left accessible) in R if there exists a positive integer n such that A is n-accessible (left n-accessible) in R.*

Proposition 1. *Let $\mathcal{M}$ be a nonempty homomorphically closed class of rings.*

(i) [16, Lemma 1] $0 \neq R \in l(\mathcal{M})$ if and only if every non-zero homomorphic image of R contains a non-zero accessible subring in $\mathcal{M}$.

(ii) [7, Lemma 3] If $0 \neq R \in ls(\mathcal{M})$, then R contains a non-zero left accessible subring in $\mathcal{M}$.

The lower radical (resp. the lower left strong radical) determined by a class $\mathcal{M}$ can be described by usage of Kurosh's chain $\{\mathcal{M}_\alpha\}$ (resp. left Kurosh's chain $\{\mathcal{M}^\alpha\}$).

Let $\mathcal{M}$ be a nonempty homomorphically closed class of rings. Define $\mathcal{M}_1 = \mathcal{M}^1 = \mathcal{M}$ and for any ordinal number $\alpha > 1$ define $\mathcal{M}_\alpha$ (resp. $\mathcal{M}^\alpha$) to be the class of all rings R such that every non-zero homomorphic image of R contains a non-zero ideal (resp. left ideal) belonging to $\mathcal{M}_\beta$ (resp. $\mathcal{M}^\beta$) for some $\beta < \alpha$.

Proposition 2. *[7] Let $\mathcal{M}$ be a nonempty homomorphically closed class of rings. Then*

$$l(\mathcal{M}) = \bigcup_{\alpha \geq 1} \mathcal{M}_\alpha, \quad ls(\mathcal{M}) = \bigcup_{\alpha \geq 1} \mathcal{M}^\alpha.$$

From the definitions of the classes $\mathcal{M}_\alpha$ and $\mathcal{M}^\alpha$ it follows:

Proposition 3. *Let $\mathcal{M}$ be a homomorphically closed class of rings. Then*
(i) $\mathcal{M}_\alpha \subseteq \mathcal{M}^\alpha$ for every ordinal α,
(ii) the classes $\mathcal{M}_\alpha$ and $\mathcal{M}^\alpha$ are homomorphically closed for every ordinal α,
(iii) if $\beta < \alpha$ then $\mathcal{M}_\beta \subseteq \mathcal{M}_\alpha$ and $\mathcal{M}^\beta \subseteq \mathcal{M}^\alpha$.

The following proposition collects some well-know properties of the classes $\mathcal{M}_\alpha$ and $\mathcal{M}^\alpha$:

Proposition 4. *(i) [7] If $0 \neq R \in \mathcal{M}^n$ for $n \geq 2$, then R contains a non-zero left $n-1$-accessible subring in $\mathcal{M}$;*
(ii) [16] $0 \neq R \in \mathcal{M}_n$ for $n \geq 2$ if and only if every non-zero homomorphic image of R contains a non-zero $n-1$-accessible subring in $\mathcal{M}$;
(iii) [16] if $A \in \mathcal{M}$ is an n-accessible subring in R then $A_R \in \mathcal{M}_n$.

Let γ be an ordinal. We say Kurosh's chain $\{\mathcal{M}_\alpha\}_{\alpha \geq 1}$ (resp. $\{\mathcal{M}^\alpha\}_{\alpha \geq 1}$) stabilises at step γ, if $\mathcal{M}_\beta = \mathcal{M}_\gamma$ (resp. $\mathcal{M}^\beta = \mathcal{M}^\gamma$) for every ordinal $\beta \geq \gamma$; which is equivalent to the condition: $l(\mathcal{M}) = \mathcal{M}_\gamma$ (resp. $ls(\mathcal{M}) = \mathcal{M}^\gamma$). We say that Kurosh's chain $\{\mathcal{M}_\alpha\}_{\alpha \geq 1}$ (resp. $\{\mathcal{M}^\alpha\}_{\alpha \geq 1}$) stabilises at exact γ-step, if $l(\mathcal{M}) = \mathcal{M}_\gamma \neq \mathcal{M}_\beta$ (resp. $ls(\mathcal{M}) = \mathcal{M}^\gamma \neq \mathcal{M}^\beta$) for every ordinal $\beta < \gamma$.

From Proposition 1(i), Proposition 2 and Proposition 4(ii) it follows that Kurosh's chain determined by a homomorphically closed class of rings stabilises at ω_0.

However, the problem if left Kurosh's chain determined by a homomorphically closed class of rings stabilises at ω_0, is still not solved [13, Question 5].

Definition 3. *(i) A class $\mathcal{M}$ of rings is called hereditary if $R \in \mathcal{M}$ and I being an ideal of R implies $I \in \mathcal{M}$.*
(ii) A homomorphically closed class $\mathcal{M}$ of rings is called powerly hereditary if for every $R \in \mathcal{M}$ there is an integer $n \geq 2$ such that $R^n \in \mathcal{M}$.

For a homomorphically closed class $\mathcal{M}$ of rings, the class $\{R \in \mathcal{M} : R^2 = 0\}$ will be denoted by $\mathcal{M}^0$. The class $\mathcal{M}^0$ is homomorphically closed as well.

In [16] it was observed that $l(\mathcal{M}) = \mathcal{M}_2$, if $\mathcal{M}$ is a homomorphically closed class of rings, which contains all zero rings and which is hereditary (Theorem 2). In [16], it was also given an example of a class $\mathcal{M} = \mathcal{M}^0$, for which $l(\mathcal{M}) = \mathcal{M}_3 \neq \mathcal{M}_2$. Then, Armendariz and Leavitt showed in [5] that $l(\mathcal{M}) = \mathcal{M}_3$ if $\mathcal{M}$ is a hereditary and homomorphically closed class of rings. In [8] and [15] it was observed that if $\mathcal{M}$ consists only of nilpotent rings then $l(\mathcal{M}) = \mathcal{M}_3$. Moreover, in [12] it was shown that if every ring R in $\mathcal{M}$ satisfies one of the following conditions: R is nilpotent or $R = R^2$ or $\mathcal{M}$ contains all the non-nilpotent ideals of R, then $l(\mathcal{M}) = \mathcal{M}_3$.

It can be seen that each of hereinabove mentioned classes of rings is powerly hereditary, so that the following proposition is a common generalisation of presented results.

Theorem 1. *Kurosh's chain determined by a homomorphically closed powerly hereditary class $\mathcal{M}$ of rings stabilises at step 3.*

Proof. By Proposition 4(ii) it is enough to prove that if in a non-zero ring $R \in l(\mathcal{M})$ there exists a non-zero accessible subring $B \in \mathcal{M}$, then in R there exists a non-zero 2-accessible subring $A \in \mathcal{M}$. By Andrunakievich's Lemma [9, Lemma 1.2.7], $B_R^r \subseteq B$ for some positive integer r. Hence, $B_R^n \subseteq B$ for every positive integer $n \geq r$.

If the ring B is nilpotent then B_R is nilpotent. Therefore $B_R^k \neq 0$ and $B_R^{k+1} = 0$ for some positive integer k. Of course, every subgroup of the additive group $(B_R^k)^+$ is an ideal in B_R. So, if $k = 1$ then $0 \neq B \in \mathcal{M}$ is a searched 2-accessible subring of R. So, it can be assumed that $k > 1$. Then $B_R^k = B_R^{k-1}R^*BR^* = B_R^{k-1}BR^*$, where R^* is a ring R, when R has a unity or a ring created from R by adjoining unity. Therefore $xBy \neq 0$ for some $x \in B_R^{k-1}$ and $y \in R^*$. Since $A = xBy$ is a subgroup of $(B_R^k)^+$, so $A \lhd B_R \lhd R$ and A is a 2-accessible subring of R. Let $f\colon B \to A$ be a transformation defined by $f(b) = xby$ for $b \in B$. Of course, f is an epimorphism of additive groups. Moreover for $b, c \in B$, $f(bc) = xbcy \in B_R^{k+1}$, so $f(bc) = 0$. Similarly, $f(b)f(c) = xbyxcy \in B_R^{2k} \subseteq B_R^{k+1}$, so $f(b)f(c) = 0$. It follows that A is a homomorphic image of B, so $A \in \mathcal{M}$.

Now, suppose that $B^n \neq 0$ for every positive integer n. The class $\mathcal{M}$ is powerly hereditary, so $B^m \in \mathcal{M}$ for some positive integer $m \geq 2$. Analogously, $(B^m)^l \in \mathcal{M}$ for some positive integer l. Hence, for every positive integer s, there exist positive integers $m_1, m_2, \ldots, m_s$ greater than 1 such that $B^t \in \mathcal{M}$ for $t = m_1 \cdot m_2 \cdot \ldots \cdot m_s$. If $2^s \geq r$ then $B^t \in \mathcal{M}$ and $B_R^t \subseteq B$. Therefore, $0 \neq B^t \lhd B_R^t \lhd R$ and it is enough to take $A = B^t$.

So that, examples of powerly hereditary classes can be created easily. Namely, having a ring R, the class $\mathcal{M}$ can be created. It consists of all homomorphic images of rings R^n for $n = 1, 2, \ldots$. Then, $\mathcal{M}$ is a homomorphically closed powerly hereditary class. If $R = 2\mathbb{Z}$ then $\mathcal{M}$ is not hereditary and it is not a class of the type described before Theorem 1. Note that $\mathbb{Z}_2^0 \in \mathcal{M}$, because $\mathbb{Z}_2^0 \cong 2\mathbb{Z}/4\mathbb{Z}$. Moreover, $\mathbb{Z}_2 \notin \mathcal{M}$, otherwise $\mathbb{Z}_2 \cong 2^k\mathbb{Z}/2^kn\mathbb{Z}$ for some positive integers k, n. But then $2 \cdot (2^k\mathbb{Z}/2^kn\mathbb{Z}) = \{0\}$, from where $2^kn | 2^{k+1}$ and $n = 2$. Therefore $(2^k\mathbb{Z}/2^kn\mathbb{Z})^2 = \{0\}$, a contradiction. Let A be the $\mathbb{Z}_2$-Zassenhaus algebra (see for instance [9, p 60]). It is well known that $A \in l(\{\mathbb{Z}_2\} \cup \{0\})$. But $\mathbb{Z}_2^0 \in \mathcal{M}$, so $A \in l(\mathcal{M})$. Suppose $A \in \mathcal{M}_2$. Then there is a non-zero ideal I of the ring A belonging to the class $\mathcal{M}$. Therefore $2I = \{0\}$ and the additive group I^+ of the ring I is cyclic, so $I^+ \cong \mathbb{Z}_2^+$. But $\mathbb{Z}_2 \notin \mathcal{M}$, thus $I \cong \mathbb{Z}_2^0$. Consequently $I = \{0, a\}$ for some non-zero $a \in A$ such that $a^2 = 0$. By properties of Zassenhaus algebras, there is $x \in A$ such that $xa \notin \{0, a\}$, which contradicts $I \lhd A$. Therefore, by Theorem 1, $l(\mathcal{M}) = \mathcal{M}_3 \neq \mathcal{M}_2$.

Now we turn to the problem of stabilisation of left Kurosh's chains for powerly hereditary classes. In [7] it was proved that if $\mathcal{M}$ is a hereditary class and $\mathcal{M}$ contains all zero rings, then $ls(\mathcal{M}) = \mathcal{M}^{\omega_0}$. E. R. Puczyłowski empowered this result by proving that if a class $\mathcal{M}$ is hereditary, then $ls(\mathcal{M}) = \mathcal{M}^4$ [13, Corollary 3.4(i)]. In his proof, the fact that $l(\mathcal{M})$ is a hereditary class, if a class $\mathcal{M}$ is hereditary, is used (cf. [11]). By using the ideas of E. R. Puczyłowski, it can be proved that $ls(\mathcal{M}) = \mathcal{M}^4$, if $\mathcal{M}$ is a powerly hereditary class such that $\mathcal{M}^0$ is a hereditary class. In the proof the following lemmas will be needed.

Lemma 1 ([14]). *For every radical $\mathcal{R}$ and for every ring R if $R/R^2 \in \mathcal{R}$ then for every positive integer n, $R^n/R^{n+1} \in \mathcal{R}$.*

Lemma 2. *Let $\mathcal{R}$ be a radical class and let $A \in \mathcal{R}$. If $A^s \in \mathcal{R}$ for some positive integer $s > 1$, then $A^k \in \mathcal{R}$ for every $k = 1, 2, \ldots, s$.*

Proof. By the assumption $A \in \mathcal{R}$, thus $A/A^2 \in \mathcal{R}$. Therefore, by Lemma 1, $A^k/A^{k+1} \in \mathcal{R}$ for $k = 1, 2, \ldots$. Thus $A^s \in \mathcal{R}$ and $A^{s-1}/A^s \in \mathcal{R}$, so $A^{s-1} \in \mathcal{R}$. Therefore $A^{s-2}/A^{s-1} \in \mathcal{R}$ and $A^{s-1} \in \mathcal{R}$, whence $A^{s-1} \in \mathcal{R}$. Proceeding further this way, we obtain $A^k \in \mathcal{R}$ for every $k = 1, 2, \ldots, s$.

From Lemma 2 we immediately obtain the following lemma.

Lemma 3. *If $\mathcal{M}$ is a powerly hereditary class of rings and $A \in \mathcal{M}$, then $A^n \in l(\mathcal{M})$ for every $n = 1, 2, \ldots$.*

The following proposition is a criterion describing when a radical class is powerly hereditary:

Proposition 5. *A radical $\mathcal{R}$ is powerly hereditary if and only if $R^2 \in \mathcal{R}$ for every $R \in \mathcal{R}$.*

Proof. $'' \Rightarrow ''$ Let $R \in \mathcal{R}$. The class $\mathcal{R}$ is powerly hereditary, so $R^m \in \mathcal{R}$ for some integer $m \geq 2$. By Lemma 2, $R^2 \in \mathcal{R}$.
$'' \Leftarrow ''$ By Definition 3.

Proposition 6. *(cf. [13, Proposition 3.3]) If a radical $\mathcal{R}$ is powerly hereditary then $ls(\mathcal{R}) = \mathcal{R}^2$ and the class $ls(\mathcal{R})$ is powerly hereditary.*

Proof. It is enough to prove that $\mathcal{R}^3 \subseteq \mathcal{R}^2$. By Proposition 3(ii) the class $\mathcal{R}^3$ is homomorphically closed, therefore the result will be true, if it is proved that for every ring $0 \neq R \in \mathcal{R}^3$ there is $0 \neq P <_l R$ such that $P \in \mathcal{R}$. Let $0 \neq R \in \mathcal{R}^3$. There is $0 \neq L <_l R$ such that $L \in \mathcal{R}^2$ and there is $0 \neq K <_l L$ such that $K \in \mathcal{R}$. Moreover, $LK <_l R$ and $LK \lhd K$. Thus, $LK/K^2 = \sum_{l \in L}(lK + K^2)/K^2$ and every ring $(lK + K^2)/K^2$ is a homomorphic image of the ring K by using transformation $x \mapsto lx + K^2$. Hence, LK/K^2 is a sum of ideals from $\mathcal{R}$, so $LK/K^2 \in \mathcal{R}$. Moreover by Proposition 5, $K^2 \in \mathcal{R}$ and therefore $LK \in \mathcal{R}$. If $LK \neq 0$, it is enough to take $P = LK$. Suppose $LK = 0$ for every $K <_l L$ such that $K \in \mathcal{R}$. Then $K^2 = 0$, $K + KL \lhd L$ and $(K + KL)^2 = 0$. Hence for $l \in L$, $Kl \lhd K + KL$ and transformation $f\colon K \to Kl$ defined by the formula $f(x) = x \cdot l$ for $x \in K$ is a homomorphism of the ring K onto the ring Kl. But $K + KL = K + \sum_{l \in L} Kl$ so $K + KL \in \mathcal{R}$. Therefore if $K <_l L$ and $K \in \mathcal{R}$, then $K \subseteq \mathcal{R}(L)$. If $L \neq \mathcal{R}(L)$ then $0 \neq L/\mathcal{R}(L) \in \mathcal{R}^2$ and there is $0 \neq M/\mathcal{R}(L) <_l L/\mathcal{R}(L)$ such that $M/\mathcal{R}(L) \in \mathcal{R}$. Then, $M \in \mathcal{R}$ and $M <_l L$, so $M \subseteq \mathcal{R}(L)$, a contradiction. Therefore $L = \mathcal{R}(L)$ and it is enough to take $P = L$.

On the contrary, suppose that the radical $\mathcal{S} = ls(\mathcal{R})$ is not powerly hereditary. Then, by ADS-Theorem [9, Theorem 3.1.2] there is $A \in \mathcal{S}$ such that $A^2 \neq 0$ and $\mathcal{S}(A^2) = 0$. The radical $\mathcal{S}$ is left strong, thus the ring A^2 has no non-zero left ideals belonging to $\mathcal{R}$. Let $L \in \mathcal{R}$ be such that $L <_l A$. Then $L^2 \in \mathcal{R}$ and $L^2 <_l A^2$, so $L^2 = 0$. Hence the ring LA is a sum of its ideals of the form La for $a \in A$ and the ring La is a homomorphic image of the ring L by using transformation $x \mapsto xa$ for $x \in L$. Thus $LA \in \mathcal{R}$. But $LA \lhd A^2$, so $LA = 0$, hence $L \lhd A$ and $L \subseteq \mathcal{R}(A)$. Therefore, $\mathcal{R}(A)$ is a greatest left ideal of the ring A belonging to the class $\mathcal{R}$ and $\mathcal{R}(A)^2 = 0$. Further, $A^2 \neq 0$, so $\mathcal{R}(A) \neq A$ and $0 \neq A/\mathcal{R}(A) \in \mathcal{S}$. Therefore, there is $K <_l A$ such that $0 \neq K/\mathcal{R}(A) \in \mathcal{R}$. Then $K \in \mathcal{R}$, hence $K \subseteq \mathcal{R}(A)$, a contradiction.

Lemma 4. *For every radical $\mathcal{R}$, $\overline{\mathcal{R}} = \{R : \text{ if } R^2 \subseteq J \lhd R \text{ then } J \in \mathcal{R}\}$ is a radical.*

Proof. The class $\overline{\mathcal{R}}$ is nonempty, because $0 \in \overline{\mathcal{R}}$. Moreover $\overline{\mathcal{R}} \subseteq \mathcal{R}$.

Let $R \in \overline{\mathcal{R}}$, $I \lhd R$ and let $(R/I)^2 \subseteq J \lhd R/I$. Then $J = A/I$ for some $A \lhd R$, such that $R^2 \subseteq A$. Therefore $A \in \mathcal{R}$ and $J \in \mathcal{R}$.

Let $I \lhd R$, $I \in \overline{\mathcal{R}}$ and $R/I \in \overline{\mathcal{R}}$. It will be proved that $R \in \overline{\mathcal{R}}$. Let $R^2 \subseteq J \lhd R$. Then $(R/I)^2 \subseteq (J+I)/I \lhd R/I$, so $(J+I)/I \in \mathcal{R}$. But $(J+I)/I \cong J/(J \cap I)$ consequently $J/(J \cap I) \in \mathcal{R}$. Moreover, $I^2 \subseteq J \cap I \lhd I$, so $J \cap I \in \mathcal{R}$. Hence $J \in \mathcal{R}$.

Let $\{I_\alpha\}$ be a chain of ideals of some ring R within the class $\overline{\mathcal{R}}$. It will be proved that $I = \bigcup I_\alpha \in \overline{\mathcal{R}}$. Let $I^2 \subseteq J \lhd I$. Then $J = \bigcup (J \cap I_\alpha)$ and for every α, $I_\alpha^2 \subseteq J \cap I_\alpha \lhd I_\alpha$. Hence, $J \cap I_\alpha \in \mathcal{R}$ for every α and $J \in \mathcal{R}$.

Therefore, $\overline{\mathcal{R}}$ is a radical.

Proposition 7. *Let $\mathcal{M}$ be a class of rings. If the class $\mathcal{M}$ is powerly hereditary and $\mathcal{M}^0$ is hereditary, then the class $l(\mathcal{M})$ is powerly hereditary.*

Proof. Let $R \in \mathcal{M}$. Then by Lemma 3, $R^2 \in l(\mathcal{M})$. Let $R^2 \subseteq I \lhd R$. Then $R/R^2 \in \mathcal{M}^0$. The class $\mathcal{M}^0$ is hereditary, hence $I/R^2 \in \mathcal{M}^0 \subseteq l(\mathcal{M})$. Now it follows that $I \in l(\mathcal{M})$ and $\mathcal{M} \subseteq \overline{l(\mathcal{M})}$. Therefore $l(\mathcal{M}) \subseteq l(\overline{l(\mathcal{M})})$. By Lemma 4 the class $\overline{l(\mathcal{M})}$ is radical, therefore $l(\overline{l(\mathcal{M})}) = \overline{l(\mathcal{M})}$. Moreover $\mathcal{M} \subseteq \overline{l(\mathcal{M})} \subseteq l(\mathcal{M})$, so finally $l(\mathcal{M}) = \overline{l(\mathcal{M})}$ and $l(\mathcal{M})$ is a powerly hereditary class.

Theorem 2. *Let $\mathcal{M}$ be a class of rings. If $\mathcal{M}$ is powerly hereditary and $\mathcal{M}^0$ is hereditary, then $ls(\mathcal{M}) = \mathcal{M}^4$ and the class $ls(\mathcal{M})$ is powerly hereditary.*

Proof. By Theorem 1, $l(\mathcal{M}) = \mathcal{M}_3$. By Proposition 7, the class $l(\mathcal{M})$ is powerly hereditary, consequently by Proposition 6, $ls(l(\mathcal{M})) = (l(\mathcal{M}))^2$. But $\mathcal{M} \subseteq l(\mathcal{M}) \subseteq ls(\mathcal{M})$, so $ls(\mathcal{M}) = ls(l(\mathcal{M}))$ and by Proposition 6, $ls(\mathcal{M})$ is powerly hereditary. Therefore $ls(\mathcal{M}) = (l(\mathcal{M}))^2$. Moreover, $\mathcal{M}_3 \subseteq \mathcal{M}^3$, so $ls(\mathcal{M}) \subseteq (\mathcal{M}^3)^2 = \mathcal{M}^4$. Hence $ls(\mathcal{M}) = \mathcal{M}^4$.

The following example shows that there is a non-hereditary homomorphically closed class $\mathcal{M}$ of rings which is powerly hereditary and for which Kurosh's chain determined by $\mathcal{M}$ stabilises exactly at step 3, and its left Kurosh's chain stabilises at step 4, so at a next step.

Example 1. [13, see Example 3.6] Let $\mathcal{M}$ be the class of rings which are homomorphic images of the ring $\mathbb{Q}[x] \oplus \mathbb{Z}^0$. Obviously, the class $\mathcal{M}$ is homomorphically closed and powerly hereditary and the class $\mathcal{M}^0$ is hereditary. The class $\mathcal{M}$ is not hereditary, because $x\mathbb{Q}[x] \notin \mathcal{M}$. By Theorem 2, $ls(\mathcal{M}) = \mathcal{M}^4$ and by Theorem 1, $l(\mathcal{M}) = \mathcal{M}_3$. Obviously $\mathbb{Q} \in \mathcal{M}$, because $\mathbb{Q} \cong \mathbb{Q}[x]/x\mathbb{Q}[x]$. Moreover $\mathbb{Q}^0 \in \mathcal{M}^2 \setminus \mathcal{M}$, $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & 0 \end{bmatrix} \in \mathcal{M}^3 \setminus \mathcal{M}^2$ and $M_2(\mathbb{Q}) \in \mathcal{M}^4 \setminus \mathcal{M}^3$, because every proper left ideal of the ring $M_2(\mathbb{Q})$ is isomorphic to $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & 0 \end{bmatrix}$. Therefore $ls(\mathcal{M}) = \mathcal{M}^4 \neq \mathcal{M}^3$. Similarly $\mathbb{Q}^0 \in \mathcal{M}_2 \setminus \mathcal{M}$ and $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & 0 \end{bmatrix} \in \mathcal{M}_3 \setminus \mathcal{M}_2$, therefore $l(\mathcal{M}) = \mathcal{M}_3 \neq \mathcal{M}_2$.

Question 1. Can we omit, in Theorem 2, the assumption that the class $\mathcal{M}^0$ is hereditary?

Question 2. Let $\mathcal{S}, \mathcal{R}$ be powerly hereditary radicals. Is the radical $l(\mathcal{S} \cup \mathcal{R})$ powerly hereditary as well?

Question 3. Supposing that the class $\mathcal{M}$ is powerly hereditary, is a class $l(\mathcal{M})$ powerly hereditary as well?

Remark 1. There are homomorphically closed classes of rings, such that they are not powerly hereditary, while the lower radicals determined by these classes are powerly hereditary. Namely, by Theorem 4.1 and Proposition 5.1 in [1], for every $n = 3, 4, \dots$ there is a homomorphically closed class $\mathcal{M}$ of rings such that $l(\mathcal{M})$ is hereditary (and therefore it is also powerly hereditary) and $l(\mathcal{M}) = \mathcal{M}_{n+1} \neq \mathcal{M}_n$. Thus class $\mathcal{M}$ is not powerly hereditary, by Theorem 1.

Now we present two examples of classes $\mathcal{M}$, for which $l(\mathcal{M})$ are powerly hereditary.

Proposition 8. *If $\mathcal{M}$ is a homomorphically closed class of nilpotent rings then $l(\mathcal{M})$ is powerly hereditary.*

Proof. Let $\mathcal{L} = l(\mathcal{M})$. Suppose that there is a ring $R \in \mathcal{L}$ such that $R^2 \notin \mathcal{L}$. By ADS-Theorem we can assume that $R^2 \neq 0$ and $\mathcal{L}(R^2) = 0$. Therefore, by Proposition 1(i), in the ring R^2 there is not a non-zero accessible subring belonging to $\mathcal{M}$. Let J be a nilpotent ideal of the ring R belonging to $\mathcal{L}$. By Lemma 2, $J^2 \in \mathcal{L}$. But $J^2 \lhd R^2$ so $J^2 = 0$. Further, the ring RJ is a sum of its ideals of the form rJ, $r \in R$ and every ideal rJ is a homomorphic image of the ring J by using transformation $j \mapsto rj$. It follows that $RJ \in \mathcal{L}$. Moreover $RJ = 0$ because $RJ \lhd R^2$. Analogously, we can show that $JR = 0$. Now it follows that $I = \sum\{J : J \in \mathcal{L}$ and J is a nilpotent ideal of $R\}$ is the largest nilpotent ideal of the ring R belonging to the class $\mathcal{L}$ and $RI = IR = 0$. Because $R^2 \neq 0$, so $I \neq R$ and $0 \neq R/I \in \mathcal{L}$. Therefore, by Proposition 1(i), in the ring R/I there is a non-zero accessible subring A/I belonging to $\mathcal{M}$. Let B/I be the ideal of the ring R/I generated by A/I. Then, by Proposition 4(iii), $B/I \in \mathcal{L}$, so

$B \in \mathcal{L}$. Moreover, $(B/I)^n \subseteq A/I$ for some positive integer n and the ring A/I is nilpotent, thus B/I is a nilpotent ideal of the ring R/I. We have $I^2 = 0$, so B is a nilpotent ideal in the ring R. Therefore $B \subseteq I$, hence $A/I = 0$, a contradiction.

Proposition 9. *Let $\mathcal{M}_0$ be a homomorphically closed class of nilpotent rings and let $\mathcal{M}_1$ be a homomorphically closed class of rings with unity. Let $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$. Then the classes $\mathcal{M}$ and $l(\mathcal{M})$ are powerly hereditary.*

Proof. Directly from the definition of the class $\mathcal{M}$ it follows that $\mathcal{M}$ is powerly hereditary. Assume the class $\mathcal{L} = l(\mathcal{M})$ is not powerly hereditary. By ADS-Theorem there is $A \in \mathcal{L}$ such that $A^2 \neq 0$ and $\mathcal{L}(A^2) = 0$. Denote $\mathcal{R} = l(\mathcal{M}_0)$ and let $I = \mathcal{R}(A)$. By Proposition 8, $I^2 \in \mathcal{R}(A)$, hence $I^2 \in \mathcal{L}$. But $I^2 \lhd A^2$, so $I^2 = 0$. Similarly as in the proof of Proposition 8 we have $AI = IA = 0$. Therefore, $0 \neq A/I \in \mathcal{L}$ and by Proposition 1(i) in the ring A/I there is a non-zero accessible subring $B/I \in \mathcal{M}$. Hence by Proposition 4(iii), $B/I \in \mathcal{M}_1$. Thus B/I is a ring with unity, so $B/I \lhd A/I$ and $B = B^2 + I$. By the lifting idempotents theorem there is $e = e^2 \in B$ such that $e + I$ is a unity of the ring B/I. Hence $B = eB \oplus r_B(e)$, where $r_B(e) = \{x \in B : ex = 0\}$. But $BI = 0$ and if $x \in r_B(e)$, then in the ring B/I we have $r + I = (e + I) \cdot (x + I) = I$, this means $x \in I$. Thus $I = r_B(e)$ and $B = eB \oplus I$. Hence $eB \cong B/I$, then $eB \in \mathcal{M}_1$. But $eB \subseteq B^2$, hence by modularity of the lattice of subgroups of the group B^+, $B^2 = (eB) \oplus (I \cap B^2)$. Moreover $B^2 = eB^2 \oplus IB = eB^2$, thus $eB^2 = (eB) \oplus (I \cap B^2)$, hence $I \cap B^2 = 0$ and $B^2 = eB$. Therefore $B^2 \in \mathcal{M}_1$. But $B^2 \lhd A^2$, so $B^2 = 0$, thus $B = I$, a contradiction.

Remark 2. By the proof of Theorem 2 it follows that if classes $\mathcal{M}$ and $l(\mathcal{M})$ are powerly hereditary, then the class $ls(\mathcal{M})$ is powerly hereditary as well, and $ls(\mathcal{M}) = \mathcal{M}^4$. On the other hand, Proposition 9 allows to construct a powerly hereditary class $\mathcal{M}$ such that $\mathcal{M}^0$ is not hereditary, but $l(\mathcal{M})$ is hereditary.

References

1. R. R. Andruszkiewicz, E. R. Puczyłowski, Accessible subrings and Kurosh's chains of associative rings, *Algebra Colloquium* 4:1 (1997), 79–88.
2. R. R. Andruszkiewicz, E. R. Puczyłowski, Kurosh's chains of associative rings, *Glasgow Math. J.* 32 (1990) 67–69.
3. R. R. Andruszkiewicz, M. Sobolewska, Accessible subrings and Kurosh's chains of associative rings, *J. Aust. Math. Soc.* 95 (2013), no. 2, 145–157.
4. R. R. Andruszkiewicz, M. Sobolewska, On the stabilisation of one-sided Kurosh's chains, *Bull. Aust. Math. Soc.* 86 (2012), no. 3, 473–480.
5. E. P. Armendariz, W. G. Leavitt, The hereditary property in the lower radical construction, *Canad. J. Math.* 20 (1968), 474–476.
6. K. I. Beidar, A chain of Kurosh may have an arbitrary finite length, *Czechoslovak Math. J.* 32 (107) (1982), no. 3, 418–422.
7. N. Divinsky, J. Krempa, A. Suliński, Strong radical properties of alternative and associative rings, *J. Algebra* 17 (1971), 369–388.
8. B. J. Gardner, Radicals of abelian groups and associative rings, *Acta Math. Acad. Sci. Hung.*, 24 (1973), 259–268.

9. B. J. Gardner, R. Wiegandt, *Radical Theory of Rings*, Marcel Dekker, Inc. 2009.
10. A. G. Heinicke, A note on lower radical constructions for associative rings, *Canad. Math. Bull.* 11 (1968), no. 1, 23–30.
11. A. E. Hoffman, W. G. Leavitt, Properties inherited by the lower radical, *Port. Math.*, 27 (1968), 63–66.
12. S. X. Liu, Y. L. Luo, A. P. Tang, J. Xiao, J. Y. Guo, Some results on modules and rings, *Publ. Soc. Math. Belg. Ser, B,* 39 (1987), 181–193.
13. E. R. Puczyłowski, On questions concerning strong radicals of associative rings, *Quaestiones Mathematicae,* 10 (1987), 321–338.
14. A. D. Sands, Radicals and one-sided ideals, *Proc. Roy. Soc. Edinburgh Sect. A* 103 (1986), No. 3, 241-251.
15. P. N. Stewart, On the lower radical construction, *Acta Math. Acad. Sci. Hung.* 25 (1974), 31-32.
16. A. Suliński, R. Anderson, N. Divinsky, Lower Radical Properties for Associative and Alternative Rings, *Journal London Math. Soc.*, 41 (1966), 417–424.

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On TI-groups

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Abstract. An abelian group $(A, +, 0)$ is called a TI-group, if every associative ring with additive group A is filial. This paper presents results concerning the structure of TI-groups. The structure theorem describing torsion TI-groups is proved, and the structure of the torsion part of mixed TI-groups is described. Furthermore, it is proved that every abelian torsion-free group of rank one is a TI-group. Numerous examples of TI-groups are given.

1 Introduction

Shalom Feigelstock studied additive groups of rings whose all subrings are two-sided ideals. Such groups are called SI*-groups*. S. Feigelstock published his results in [9]. In [7] we noted and corrected some inconsistencies in Feigelstock's article and we presented new results concerning the structure of SI-groups. In particular, we introduced there the necessary new terminology and symbols allowing to distinguish between the both kinds of the ring structures on abelian groups: associative and general.

Associative rings in which all subrings are two-sided ideals, play a significant role in the ring theory. Such rings are called hamiltonian rings or H-rings, because these structures are somewhat analogous to the hamiltonian groups. H-rings were systematically studied by many authors. The most valuable results were achieved by L. Rédei, R. L. Kruse and V. I. Andrijanov (cf. [16], [15], [1]). For this reason, an abelian group A such that every associative ring with additive group A is an H-ring is called an SI_H*-group*.

A natural generalization of H-rings are filial rings (i.e. associative rings in which every accesible subring is an ideal). Filial rings were studied by G. Ehrlich, A. D. Sands, M. Filipowicz, E. R. Puczyłowski, R. R. Andruszkiewicz and K. Pryszczepko (cf. [8], [17], [10], [11], [12], [6], [2], [4], [3], [5]).

In this context, it is interesting to investigate the structure of abelian groups A such that every associative ring with additive group A is filial. Such groups are called TI*-groups* (from: *transitive ideals*). In this paper we present results concerning the structure of TI-groups. We prove the structure theorem describing torsion TI-groups and we describe the structure of the torsion part of mixed TI-groups. Surprisingly, it is identical to the structure of torsion part of mixed SI_H-groups. Moreover, we prove that every abelian torsion-free group of rank one is a TI-group. We give numerous examples of TI-groups.

Symbols $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ stand for the field of rationals, ring of integers, the set of all prime numbers and the set of all positive integers, respectively. The least common multiple of two integers k and l is denoted by $LCM(k,l)$. The exponent of a group A is denoted by $\omega(A)$. If $\{A_i : \ i \in I\}$, where I is a nonempty set, is a family of groups and $x \in \bigoplus_{i\in I} A_i$, then the support of x is denoted by $supp(x)$. In this paper, only the abelian groups with a traditionally additive notation applied for them will be considered. For a given positive integer n, $Z(n)$ denotes the cyclic group of order n. By $\mathbb{Z}_n$ we denote the ring $\mathbb{Z}/n\mathbb{Z}$. Symbol R^+ stands for the additive group of the ring R. A two-sided ideal I of a ring R is denoted by $I \lhd R$. If a is an element of the ring R, then symbols $[a]$, $\langle a\rangle$, $o(a)$ stand for the ring generated by a, the cyclic subgroup of R^+ generated by a and the order of a in the group R^+, respectively. Every abelian group $(A,+,0)$ can be provided with a ring structure in a trivial way by defining $a \cdot b = 0$, for all $a,b \in A$. Such a ring is called a *zero-ring* and it is denoted by A^0. All other designations are consistent with generally accepted standards.

For preliminary knowledge of divisible groups and tensor product of abelian groups we refer the reader to [13] and [14].

2 Preliminaries

2.1 Definitions and notations

For an arbitrary abelian group $(A,+,0)$ and a prime number p we define a p-component A_p of the group A:

$$A_p = \{a \in A : \ p^n a = 0, \text{ for some } n \in \mathbb{N}\}.$$

Often we will use the designation:

$$\mathbb{P}(A) = \{p \in \mathbb{P} : \ o\left(a\right) = p, \text{ for some } a \in A\}.$$

The torsion part of A is denoted by $T(A)$. Of course, $T(A) = \bigoplus_{p\in\mathbb{P}(A)} A_p$.

We remind the reader that a subring S of a ring R is said to be n-accessible in R, if there exist subrings $R = S_0, S_1, \ldots, S_{n-1}, S_n = S$ of R, such that $S_i \lhd S_{i-1}$ for $i = 1, 2, \ldots, n$. We say that a subring S is accessible in R if it is n-accessible for some positive integer n.

Definition 1. *An associative ring R is said to be filial if every accessible subring of R is an ideal. Of course, the ring R is filial if and only if it satisfies: if $J \lhd I \lhd R$, then $J \lhd R$.*

Definition 2. *An abelian group A is called a TI-group, if every ring R with $R^+ = A$ is filial.*

Remark 1. Every SI_H-group is a TI-group.

2.2 Some useful facts about filial rings and *H*-rings

Example 1. It is easy to check that every ring of cardinality p^2 is filial for an arbitrary prime number p. In fact, if S and T are subrings of a ring R of cardinality p^2 satisfying $S \lhd T \lhd R$, then we need only consider three cases: $|T| = p^2$, $|T| = p$ and $|T| = 1$. In each of these cases, we obtain $S \lhd R$. Therefore every group of order p^2 is a TI-group, for each prime number p.

Proposition 1. *Let p be a prime number and let A be an abelian group such that $A \neq T(A)$. Then the ring $R = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 0 \end{pmatrix} \oplus A^0$ is not filial.*

Proof. Take any $a \in A$ such that $o(a) = \infty$. Define:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\alpha = (X, a)$. Since $X^2 = 0$ and $a^2 = 0$, it follows that $\alpha^2 = 0$, and consequently $[\alpha] = \langle \alpha \rangle$. Moreover, $(kX, b)(lX, c) = (lX, c)(kX, b) = (kX, b)(tY, d) = (0, 0) \in [\alpha]$, for all $k, l, t \in \{0, 1, \ldots, p-1\}$, $b, c, d \in A$, and $(Y, 0)(X, a) = (X, 0) \notin [\alpha]$. Thus:

$$[\alpha] \lhd \begin{pmatrix} 0 & \mathbb{Z}_p \\ 0 & 0 \end{pmatrix} \oplus A^0 \lhd R \text{ and } [\alpha] \ntriangleleft R.$$

Therefore the ring R is not filial.

Proposition 2. *Let p and n be a prime number and a positive integer, respectively. Let A be an abelian group such that $A_p \neq \{0\}$ or $T(A) \neq A$. If $n > 1$, then the ring $R = \mathbb{Z}_{p^n} \oplus A^0$ is not filial.*

Proof. Take any $a \in A$. Let $\alpha = (p^{n-1}, a)$. Then $\alpha^2 = 0$, so $[\alpha] = \langle \alpha \rangle$. Hence by the commutativity of the ring R we obtain $[\alpha] \lhd R\alpha + [\alpha] \lhd R$. Suppose, contrary to our claim, that $(1, 0)\alpha \in [\alpha]$. Then there exists $k \in \mathbb{Z}$ such that $(p^{n-1}, 0) = k(p^{n-1}, a)$. Hence $p^{n-1} = kp^{n-1}$ and $0 = ka$. If $o(a) = \infty$, then from the equality $ka = 0$ it follows that $k = 0$. Thus $p^{n-1} = 0$ in $\mathbb{Z}_{p^n}$, a contradiction. If $o(a) = p$, then $k = lp$, for some $l \in \mathbb{Z}$. Therefore $p^{n-1} = kp^{n-1} = lp^n = 0$ in $\mathbb{Z}_{p^n}$, a contradiction. Thus $(1, 0)\alpha \notin [\alpha]$. Therefore $[\alpha] \ntriangleleft R$, and consequently R is not filial.

Corollary 1. *If m and n are positive integers satisfying $m \geq n$ and $m > 1$, then $R = \mathbb{Z}_{p^m} \oplus Z(p^n)^0$ is not a filial ring.*

The reader will be able to find following facts about H-rings in [7], but we present this results for the transparency of our paper.

Proposition 3. *If A is an H-ring satisfying $A = pA$, for some $p \in \mathbb{P}$, then $R = Z(p)^0 \oplus A$ is an H-ring.*

Proof. Let $\alpha, \beta \in R$. Then $\alpha = (k, a)$, $\beta = (l, b)$, for some $k, l \in Z(p)^0$ and $a, b \in A$. Since $A = pA$, there exists $c \in A$ such that $b = pc$. Moreover $[a] \lhd A$, so there exist $n \in \mathbb{N}$ and $k_1, k_2, \ldots, k_n \in \mathbb{Z}$ such that $ac = \sum_{i=1}^{n} k_i a^i$. Hence $ab = a(pc) = p(ac) = p\sum_{i=1}^{n} k_i a^i$. It is clear that $\alpha^m = (0, a^m)$, for every $m \in \mathbb{N}$ such that $m \geq 2$. Notice further that $p(0, a) = (0, pa) = p(k, a) = p\alpha$. Hence $p(0, a^m) = p\alpha^m$ for all $m \in \mathbb{N}$. Therefore $\alpha\beta = (0, ab) = (0, p\sum_{i=1}^{n} k_i a^i) = \sum_{i=1}^{n} k_i(p(0, a^i)) = \sum_{i=1}^{n}(pk_i)\alpha^i \in [\alpha]$. Thus $[\alpha] \lhd R$.

Lemma 1. *Let A be an H-ring satisfying $A = pA$, for some $p \in \mathbb{P}$. If $R = \mathbb{Z}_p \oplus A$, then $(0, a) \in [(1, a)]$, for all $a \in A$.*

Proof. Take any $a \in A$. Then $a = pb$, for some $b \in A$ and $[a] \lhd A$. So there exist $s \in \mathbb{N}$ and $k_1, k_2, \ldots, k_s \in \mathbb{Z}$ such that $ab = k_1 a + k_2 a^2 + \cdots + k_s a^s$. Hence $a^2 = a(pb) = p(ab) = (pk_1)a + (pk_2)a^2 + \cdots + (pk_s)a^s$. Therefore $a = (pk_1 + 1)a + (pk_2 - 1)a^2 + (pk_3)a^3 + (pk_4)a^4 + \cdots + (pk_s)a^s$. Moreover $p \mid (pk_1 + 1) + (pk_2 - 1) + pk_3 + pk_4 + \cdots + pk_s$, hence $(0, a) = (pk_1 + 1)(1, a) + (pk_2 - 1)(1, a)^2 + pk_3(1, a)^3 + pk_4(1, a)^4 + \cdots + pk_s(1, a)^s$. Thus $(0, a) \in [(1, a)]$.

Proposition 4. *If A is an H-ring satisfying $A = pA$, for some $p \in \mathbb{P}$, then $R = \mathbb{Z}_p \oplus A$ is an H-ring.*

Proof. Take any $\alpha \in R$. Then $\alpha = (k, a)$, for some $k \in \mathbb{Z}_p$, $a \in A$. If $k \neq 0$, then there exists $l \in \mathbb{Z}_p$ satisfying $kl = 1$. Since $p \nmid l$, there exist $x, y \in \mathbb{Z}$ such that $(0, a) = x(0, pa) + y(0, la)$. It is evident that $(0, pa) = p\alpha$. Moreover, $(0, la) \in [(1, la)]$, by Lemma 1, and $(1, la) = (kl, la) = l(k, a) = l\alpha$. Thus $(0, a) \in [\alpha]$. As $(k, 0) = \alpha - (0, a) \in [\alpha]$ we have $(k) \oplus [a] \subseteq [\alpha]$. Moreover $\alpha \in (k) \oplus [a]$, so $[\alpha] \subseteq (k) \oplus [a]$. Therefore $[\alpha] = (k) \oplus [a] \lhd R$.

2.3 Ring multiplication on some specific abelian groups

Remark 2. Let p be a prime number. It is a well-known fact that up to isomorphism there exist only two rings of cardinality p: the zero-ring $Z(p)^0$ and the field $\mathbb{Z}_p$. So if $s > 1$ is a square-free number, R is a ring with $R^+ = Z(s)$, and r is the product of all prime divisors p of s, for which $R_p \cong \mathbb{Z}_p$, then the ring R satisfies the condition $R^n \cong \mathbb{Z}_r$, for all $n \in \mathbb{N}$ such that $n \geq 2$.

Proposition 5. *Let A and H be abelian groups such that $A = T(A)$, $\omega(A_p) < \infty$, $H = pH$, and $H_p = \{0\}$, for all $p \in \mathbb{P}(A)$. Then every ring multiplication $*$ on the group $G = A \oplus H$ satisfies the following conditions:*

(i) $A * H = H * A = \{0\}$;
(ii) $A * A \subseteq A$;
(iii) $H * H \subseteq H$;

The proof of the above proposition will be available in [7], but we will give it for the completeness of our paper:

Proof. (i). This follows directly from the p-divisibility of H for every $p \in \mathbb{P}(A)$, and the distributivity of multiplication with respect to addition.
(ii). Take any $a_1, a_2 \in A$. Then $a_1 * a_2 = a_3 + h$, for some $a_3 \in A$, $h \in H$. Let $m = LCM\big(o(a_1), o(a_2), o(a_3)\big)$. Then $0 = m(a_1 * a_2) = m(a_3 + h) = mh$, hence $h \in T(H)$. Let $p \in \mathbb{P}$. If $p \mid o(h)$, then $p \mid m$, thus from the definition m it follows that $p \in \mathbb{P}(A)$. But $H_p = \{0\}$, for every $p \in \mathbb{P}(A)$, hence $h = 0$.
(iii). Take any $h_1, h_2 \in H$. Then $h_1 * h_2 = a + h$, for some $a \in A$, $h \in H$. So there exists a nonempty finite subset P of the set $\mathbb{P}(A)$ such that $a \in \bigoplus_{p\in P} A_p$. Moreover $n\left(\bigoplus_{p\in P} A_p\right) = \{0\}$, for $n = \prod_{p\in P} \omega(A_p)$. But $H = pH$, for every $p \in \mathbb{P}(A)$, so $h_1 = nh_1'$, $h_2 = nh_2'$ and $h = n^2h'$, for some $h_1', h_2', h' \in H$. Therefore $n^2(h_1' * h_2') = a + n^2h'$, hence $a = n^2((h_1' * h_2') - h') \in n^2G = \bigoplus_{q\in\mathbb{P}(A)\setminus P}(n^2A_q) \oplus H$. As $a \in \bigoplus_{p\in P} A_p$ we have $a = 0$. Therefore $h_1 * h_2 \in H$.

3 Main results

Proposition 6. *Let p and n be a prime number and a positive integer, respectively, and let D be a nontrivial divisible p-group. Then $A = Z(p^n) \oplus D$ is a TI-group if and only if $n = 1$.*

Proof. If A is a TI-group, then we apply Proposition 2 to obtain $n = 1$.
The opposite implication follows at once from Lemma 6 in [9] and Remark 1.

Proposition 7. *If H is an abelian group with $H_p \neq \{0\}$, for some prime number p, then $A = \mathbb{Z}_p^+ \oplus \mathbb{Z}_p^+ \oplus H$ is not a TI-group.*

Proof. Take any $h \in H$ such that $o(h) = p$. It is easy to check that the function $*\colon A \times A \to A$ defined by:

$$(k_1, l_1, h_1) * (k_2, l_2, h_2) = (0, 0, (k_1l_2 + k_2l_1)\,h)\,,$$

for all $k_1, k_2, l_1, l_2 \in \mathbb{Z}_p^+,\ h_1, h_2 \in H$, is a nonzero associative and commutative binary operation on the set A. Moreover, for all $k_i, l_i, \in \mathbb{Z}_p^+,\ h_i \in H$, where $i = 1, 2, 3$, we have:
$(k_1, l_1, h_1) * ((k_2, l_2, h_2) + (k_3, l_3, h_3)) = (k_1, l_1, h_1) * (k_2 + k_3, l_2 + l_3, h_2 + h_3) = (0, 0, (k_1(l_2+l_3)+(k_2+k_3)l_1)h) = (0, 0, (k_1l_2+k_1l_3+k_2l_1+k_3l_1)h) = (0, 0, ((k_1l_2+k_2l_1)+(k_1l_3+k_3l_1))h) = (0, 0, (k_1l_2+k_2l_1)h)+(0, 0, (k_1l_3+k_3l_1)h) = (k_1, l_1, h_1)*(k_2, l_2, h_2) + (k_1, l_1, h_1) * (k_3, l_3, h_3)$.
Therefore $R = (A, +, *, 0)$ is an associative commutative nonzero ring. Let $x = (1, 0, 0)$. Then $x^2 = 0$, so $[x] = \langle x\rangle$ and consequently $\langle x\rangle \lhd R * x + \langle x\rangle \lhd R$. As $(0, 1, 0) * x = (0, 0, h) \notin [x]$ we have $[x] \ntriangleleft R$. Therefore R is not filial. Thus A is not a TI-group.

Directly from Proposition 2 in [6] we obtain the following:

Proposition 8. *An abelian torsion group A is a TI-group if and only if A_p is a TI-group, for every prime number p.*

Proposition 9. *A direct summand of a TI-group is a TI-group.*

Proof. Let A and B be abelian groups and let $G = A \oplus B$. Suppose, contrary to our claim, that A is not a TI-group. Then there exists a ring S such that $S^+ = A$ and S is not filial. Let $R = S \oplus B^0$. Then $R^+ = G$ and $S \oplus \{0\} \lhd R$. Hence from Lemma 1 in [8], we infer that R is not filial. Therefore G is not a TI-group.

Theorem 1. *A nontrivial torsion abelian group A is a TI-group if and only if each of its nontrivial p-components A_p satisfies one of the following conditions:*

(i) $A_p = Z(p^n)$, *n a positive integer;*
(ii) $A_p = Z(p^n) \oplus D$, *with D a divisible p-group and $n = 0$ or $n = 1$;*
(iii) $A_p = Z(p) \oplus Z(p)$.

In other words, A is a TI-group if and only if either A_p is an SI_H-group or $A_p = Z(p) \oplus Z(p)$, for every $p \in \mathbb{P}(A)$.

Proof. Suppose that A is a TI-group. Take any $p \in \mathbb{P}(A)$. Then A_p is a TI-group, by Proposition 8. If A_p is an SI_H-group, then A_p satisfies either (i) or (ii) (cf. Lemma 6 in [9]). Now suppose that A_p is not an SI_H-group. Let D be a maximal divisible subgroup of A_p. It follows from Theorem 21.2 in [13] that $A_p = D \oplus B$ for some reduced subgroup B of A_p. Since A_p is not an SI_H-group, it follows that $B \neq \{0\}$. Suppose that $D \neq \{0\}$. If $B = Z(p^n)$, for some $n \in \mathbb{N}$, then $n > 1$ and A_p is not a TI-group, by Proposition 6. Hence, by Corollary 27.3 in [13], Proposition 9 and Corollary 1 we infer that $Z(p) \oplus Z(p)$ is a direct summand of B. Therefore $C = Z(p) \oplus Z(p) \oplus D$ is a direct summand of A_p. Moreover, C is a TI-group, by Proposition 9. As $D \neq \{0\}$ we have a contradiction with Proposition 7. Thus $D = \{0\}$. Since A_p is not an SI_H-group, Corollary 27.3 in [13], Proposition 9 and Corollary 1 imply that $A_p = Z(p) \oplus Z(p) \oplus E$, for some subgroup E of A_p. Combining this with Proposition 7 we infer that $E = \{0\}$, and consequently $A_p = Z(p) \oplus Z(p)$.

The opposite implication follows directly from Lemma 6 in [9], Remark 1, Example 2 and Proposition 8.

Proposition 10. *Let A be a TI-group. Then A_p is a direct summand of A, for every prime p.*

Proof. Let D be a maximal divisible subgroup of A_p. Then $A = D \oplus H$ and $A_p = D \oplus H_p$, for some subgroup H of A (cf. Theorem 21.2 in [13]). From the maximality of the group D it follows that the group H_p is reduced. If $H_p = \{0\}$, then $A_p = D$, so A_p is a direct summand of A. Now suppose that $H_p \neq \{0\}$. It follows from Corollary 27.3 in [13] that $A = D \oplus Z(p^n) \oplus K$, for some $n \in \mathbb{N}$ and some subgroup K of A. If $K_p = \{0\}$, then $A_p = D \oplus Z(p^n)$ is a direct summand of A. Now suppose that $K_p \neq \{0\}$. We apply Corollary 27.3 in [13] again to obtain $A = D \oplus Z(p^n) \oplus Z(p^m) \oplus L$, for some $m \in \mathbb{N}$ and some subgroup L of A. It follows from Proposition 9 and Corollary 1 that $m = n = 1$. Thus $A = Z(p) \oplus Z(p) \oplus (D \oplus L)$. It follows from Proposition 7 that $(D \oplus L)_p = \{0\}$, and consequently $A_p = Z(p) \oplus Z(p)$ is a direct summand of A.

Propositions 9 and 10 imply at once the following:

Proposition 11. *A p-component of a TI-group is a TI-group, for every prime number p.*

The next result follows directly from Propositions 11 and 8.

Theorem 2. *A torsion part of a TI-group is a TI-group.*

In [7] we proved a more accurate version of Theorem 10 in [9]. Namely, we obtained that if G is a mixed SI_H-group, then $T(G) = \bigoplus_{p\in\mathbb{P}(G)} Z(p)$. It turns out that this result remains true for TI-groups. The proof of this fact follows partially from the proof of Theorem 10 in [9], but we present complete proof for the transparency of our paper.

Theorem 3. *If G is a mixed TI-group, then $T(G) = \bigoplus_{p\in\mathbb{P}(G)} Z(p)$.*

Proof. Take any $p \in \mathbb{P}(G)$. Then G_p is a TI-group, by Proposition 11. Suppose, contrary to our claim, that $G_p = Z(p)\oplus Z(p)$. It follows from Proposition 10 that $G = G_p \oplus H$, for some subgroup H of the group G. Since G is mixed, $H \neq T(H)$. Therefore Proposition 1 implies that the algebraic system $(G, +, *, 0)$, where the function $*\colon G \times G \to G$ is given by the formula:

$$(k_1, l_1, a_1) * (k_2, l_2, a_2) = (k_1k_2, k_1l_2, 0),$$

for all $k_1, k_2, l_1, l_2 \in Z(p)$, $a_1, a_2 \in H$, is an associative ring, which is not filial. Hence G is not a TI-group, a contradiction. Combining this with Theorem 1 we conclude that G_p is an SI_H-group. Suppose, contrary to our claim, that $G \neq Z(p^n)$, for every $n \in \mathbb{N}$. From Lemma 6 in [9] it follows that there exists a subgroup A of G satisfying $G_p = Z(p^\infty) \oplus A$. Take any $x \in Z(p^\infty)$ and $a \in A$ such that $o(x) = p^2$ and $o(a) = \infty$. It is easy to check that the function $\psi\colon \langle a\rangle \times \langle a\rangle \to Z(p^\infty)$ defined by the formula $\psi(ka, la) = (kl)x$, for all $k, l \in \mathbb{Z}$, is bilinear. Therefore Theorem 59.1 in [13] implies the existence of a unique homomorphism $\varphi\colon \langle a\rangle \otimes \langle a\rangle \to Z(p^\infty)$ satisfying $\psi = \varphi \circ e$.

$$\begin{tikzcd}
\langle a\rangle \times \langle a\rangle \arrow[rr, "e"] \arrow[dr, "\psi"'] & & \langle a\rangle \otimes \langle a\rangle \arrow[dl, dashed, "\varphi"] \\
& Z(p^\infty) &
\end{tikzcd}$$

Let $\imath$ be the restriction of $id_{A\otimes A}$ to $\langle a\rangle \otimes \langle a\rangle$. Then $\imath$ is a monomorphism. Moreover, a group $Z(p^\infty)$ is injective by Theorem 24.5 in [13]. Thus there exists a homomorphism $\phi\colon A \otimes A \to Z\,(p^\infty)$ satisfying $\varphi = \phi \circ \imath$.

$$\begin{tikzcd}
0 \arrow[r] & \langle a\rangle \otimes \langle a\rangle \arrow[r, "\imath"] \arrow[d, "\varphi"'] & A \otimes A \arrow[dl, dashed, "\phi"] \\
& Z(p^\infty) &
\end{tikzcd}$$

Note that $\phi(a \otimes a) = (\phi \circ \imath)(a \otimes a) = \varphi(a \otimes a) = (\varphi \circ e)(a, a) = \psi(a, a) = x$. We define the function $*\colon (Z(p^\infty) \oplus A) \times (Z(p^\infty) \oplus A) \to Z(p^\infty) \oplus A$ by the formula:

$$(x_1, a_1) * (x_2, a_2) = (\phi(a_1 \otimes a_2), 0), \text{ for all } x_1, x_2 \in Z(p^\infty),\ a_1, a_2 \in A$$

As $(0, a) * (0, a) = (x, 0)$ and $o(x) = p^2$ we conclude that $*$ is a nonzero binary operation on the set $Z(p^\infty) \oplus A$. Since $(x, 0) * (y, b) = (y, b) * (x, 0) = (0, 0)$, for all $x, y \in Z(p^\infty)$ and $b \in A$, it follows that the operation $*$ is associative. Moreover, $\big((x_1, a_1) + (x_2, a_2)\big) * (x_3, a_3) = (x_1 + x_2, a_1 + a_2) * (x_3, a_3) = \big(\phi((a_1 + a_2) \otimes a_3), 0\big) = \big(\phi(a_1 \otimes a_3 + a_2 \otimes a_3), 0\big) = \big(\phi(a_1 \otimes a_3) + \phi(a_2 \otimes a_3), 0\big) = \big(\phi(a_1 \otimes a_3), 0\big) + \big(\phi(a_2 \otimes a_3), 0\big) = (x_1, a_1) * (x_3, a_3) + (x_2, a_2) * (x_3, a_3)$, for all $x_1, x_2, x_3 \in Z(p^\infty)$, $a_1, a_2, a_3 \in A$. The rest of the proof of distributivity of $*$ with respect to addition runs as before. Therefore $R = (G, +, *)$ is a nonzero associative ring. Let $\alpha = (0, pa)$. Then we obtain:
$\alpha * \alpha = \big(\phi((pa) \otimes (pa)), 0\big) = \big(\phi(p^2(a \otimes a)), 0\big) = \big(p^2\phi(a \otimes a), 0\big) = (p^2 x, 0) = (0, 0)$, hence $[\alpha] = \langle\alpha\rangle$. As $o(x) = p^2$ we have $(0, a) * \alpha = (px, 0) \neq (0, 0)$. Hence $(0, a) * \alpha \notin [\alpha]$, and consequently $[\alpha] \ntriangleleft R$. From the equality $R^3 = \{0\}$ and Proposition 1 in [6] it follows that the ring R is not filial. Therefore G is not a TI-group, a contradiction. Thus $G_p = Z(p^n)$, for some $n \in \mathbb{N}$. Since $G = G_p \oplus H$ and $H \neq T(H)$, Proposition 2 implies that $n = 1$. Thus $G_p = Z(p)$, for every $p \in \mathbb{P}(G)$.

The next two results concerning SI_H-groups have been achieved by us in [7]. We present them for the sake of Remark 1. We give complete proofs, because [7] is not yet available.

Lemma 2. *Let A be an SI_H-group satisfying $A = pA$ and $A_p = \{0\}$, for some $p \in \mathbb{P}$. Then $G = Z(p) \oplus A$ is an SI_H-group.*

Proof. Let R be an arbitrary associative ring with $R^+ = G$. Then it follows from Proposition 5 that $Z(p)^2 \subseteq Z(p)$, $A^2 \subseteq A$ and $Z(p) \cdot A = A \cdot Z(p) = \{0\}$. Thus $R = R_1 \oplus R_2$, where R_1 and R_2 are some rings such that $R_1^+ = Z(p)$ and $R_2^+ = A$. Therefore the assertion follows from Remark 2, Propositions 3 and 4.

Proposition 12. *Let $\emptyset \neq P \subseteq \mathbb{P}$ and let A be an SI_H-group satisfying $A = pA$ and $A_p = \{0\}$, for all $p \in P$. Then $G = \big(\bigoplus_{p \in P} Z(p)\big) \oplus A$ is an SI_H-group.*

Proof. If $|P| < \infty$, then the assertion follows from Lemma 2 by simple induction argument. Now suppose that $|P| = \infty$. Let R be an arbitrary associative ring with $R^+ = G$. Take any $\alpha, \beta \in R$. then $\alpha = (k, a)$, $\beta = (l, b)$, for some $k, l \in \bigoplus_{p \in P} Z(p)$ and $a, b \in A$. Let $P_{kl} = supp(k) \cup supp(l)$. There exists a subgroup H of the group G such that $H \cong \bigoplus_{p \in P_{kl}} Z(p) \oplus A$. Of course, $\alpha, \beta \in H$. Moreover $|P_{kl}| < \infty$, thus H is an SI_H-group, by first part of the proof. It follows from Proposition 5 and Remark 2 that H is a subring of the ring R. Therefore $[\alpha] \triangleleft H$, hence $\alpha\beta, \beta\alpha \in [\alpha]$. Hence by the arbitrary choice of the element β of the ring R we obtain $[\alpha] \triangleleft R$. Since α has also been chosen arbitrarily, R is an H-ring. Therefore G is an SI_H-group.

Theorem 4. *Every abelian torsion-free group of rank one is a TI-group.*

Proof. Since every abelian torsion-free group of rank one can be embedded in the group $\mathbb{Q}^+$ (cf. [14], p. 85), it is sufficient to prove the theorem only for subgroups of the group $\mathbb{Q}^+$. Let A be a subgroup of $\mathbb{Q}^+$ and let R be a ring with $R^+ = A$. It follows from Proposition 14 in [9] that there exists a subgroup B of $\mathbb{Q}^+$ such that B contains the number 1 and B is isomorphic to A. Moreover, B with the ring structure naturally induced from R is a subring of the field that is isomorphic to the field of rationals. Hence from Theorem 6 in [8] we infer that the ring B is filial. Hence R is filial too. Therefore A is a TI-group.

References

1. V. I. Andrijanov, *Periodic hamiltonian rings*, Mat. Sb. (N.S.), 74(116) (1967), 241–261; translation in Mat. Sbornik 74(116) No. 2(1967), 225–241.
2. R. R. Andruszkiewicz, *The classification of integral domains in which the relation of being an ideal is transitive.*, Comm. Algebra 31 (2003), 2067–2093.
3. R. R. Andruszkiewicz, K. Pryszczepko, *A classification of commutative reduced filial rings.*, Comm. Algebra 37 (2009), 3820–3826.
4. R. R. Andruszkiewicz, K. Pryszczepko, *The classification of commutative, noetherian, filial rings with identity*, Comm. Algebra 40(5) (2012), 1690–1703.
5. R. R. Andruszkiewicz, K. Pryszczepko, *A classification of commutative torsion filial rings.*, J. Aust. Math. Soc. 95(3) (2013), 289–296.
6. R. R. Andruszkiewicz, E. R. Puczyłowski, *On filial rings*, Portugal. Math. 45 (1988), 139–149.
7. R. R. Andruszkiewicz, M. Woronowicz, *On SI-groups*, sent paper;
8. G. Ehrlich, *Filial rings*, Portugal. Math. 42 (1983–1984), 185–194.
9. S. Feigelstock, *Additive groups of rings whose subrings are ideals*, Bull. Austral. Math. Soc. 55 (1997), 477–481.
10. M. Filipowicz, E. R. Puczyłowski, *Left filial rings*, Algebra Colloq. 11 (2004), 335–344.
11. M. Filipowicz, E. R. Puczyłowski, *On filial and left filial rings*, Publ. Math. Debrecen. 66(3-4) (2005), 257–267.
12. M. Filipowicz, E. R. Puczyłowski, *The structure of let filial algebras over the field*, Taiwanese J. Math. 13(3) (2009). 1017–1029.
13. L. Fuchs, *Infinite abelian groups volume 1*, Academic Press, New York, London, 1970.
14. L. Fuchs, *Infinite abelian groups volume 2*, Academic Press, New York, London, 1973.
15. R. L. Kruse, *Rings in which all subrings are ideals*, Canad. J. Math. 20 (1968) 862–871.
16. L. Rédei, *Vollidealringe im weiteren Sinn.* I, Acta Math. Acad. Sci. Hungar. 3 (1952), 243–268.
17. A. D. Sands, *On ideals in over-rings*, Publ. Math. Debrecen. 35 (1988), 273–279.

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Remarks on Algebraic and Geometric Properties of the Spark of a Matrix

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Abstract. We recall some known algebraic and geometric results concerning the spark of a matrix (over an arbitrary field) and provide a few new examples and observations.

1 Introduction and Preliminaries

Throughout the text m and n stand for positive integers (in other words, $m, n \in \mathbb{N}$), and $\mathbb{F}$ is a field. The elements of the space $\mathbb{F}^n$ are understood to be columns. We denote the zero vector in $\mathbb{F}^n$ by $\mathbf{0}$. Recall that the *Hamming weight* of a vector $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{F}^n$ is defined by

$$\|x\|_0 = \#\{j \in \{1, \dots, n\} : x_j \neq 0\}.$$

Finally, for a non-negative integer k we define $\Sigma_k = \{x \in \mathbb{F}^n : \|x\|_0 \leq k\}$.

We denote by $\mathcal{M}_{m\times n}(\mathbb{F})$ the vector space of all $m \times n$ matrices over $\mathbb{F}$. For a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ we define $\mathrm{Ker}(A) = \{x \in \mathbb{F}^n : Ax = \mathbf{0}\}$.

The nth full linear group over $\mathbb{F}$ will be denoted by $\mathcal{GL}_n(\mathbb{F})$.

Now, let V be a nonzero finite dimensional vector space over $\mathbb{F}$ and let $d = \dim V$. Recall that a set $E \subseteq V$ is said to be algebraic, if there exist a linear isomorphism $\varphi : V \longrightarrow \mathbb{F}^d$, a positive integer s, and polynomials $f_1, \dots, f_s \in \mathbb{F}[x_1, \dots, x_d]$ such that $E = \{v \in V : f_1(\varphi(v)) = \ldots = f_s(\varphi(v)) = 0\}$. We define the *linear capacity* of an algebraic set $E \subseteq V$ by

$$\Lambda(E) = \sup\{\dim L : L \text{ is a linear subspace of } V,\ L \subseteq E\}.$$

For more information on algebraic geometry we refer to [2].

In [1], Donoho and Elad introduced the notion of spark of a matrix.

Definition 1. *Let $C_1, \dots, C_n \in \mathbb{F}^m$ be the columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. The spark of A is defined to be the infimum of the set of all positive integers ℓ such that*

$$\exists\, j_1, \dots, j_\ell \in \{1, \dots, n\} : \begin{cases} j_1 < \ldots < j_\ell, \\ C_{j_1}, \dots, C_{j_\ell} \text{ are linearly dependent (over } \mathbb{F}). \end{cases}$$

Let us collect a few simple and well known properties of the spark.

Proposition 1. *For any matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ the following hold true:*
(i) $\operatorname{spark}(A)$ *is either a positive integer or* $+\infty$*,*
(ii) $\operatorname{spark}(A) = +\infty$ *if and only if* $\operatorname{rank}(A) = n$*,*
(iii) $\operatorname{spark}(A) = 1$ *if and only if A has a zero column,*
(iv) if $\operatorname{spark}(A) \neq +\infty$*, then* $\operatorname{spark}(A) \leq \operatorname{rank}(A) + 1$*,*
(v) $\operatorname{spark}(A) = \inf\limits_{x\in \operatorname{Ker}(A)\setminus\{\mathbf{0}\}} \|x\|_0$*,*
(vi) if $\mathbb{K}$ is an extension field of $\mathbb{F}$, then the spark of A as an element of $\mathcal{M}_{m\times n}(\mathbb{K})$ is equal to the spark of A as an element of $\mathcal{M}_{m\times n}(\mathbb{F})$.

The spark of a matrix plays a quite important role in the mathematical theory of Compressed Sensing [3]. It is also of separate interest. In the present note we recall some known results concerning the spark and provide a few examples and observations, which seem to be new or not explicitly stated in the literature.

2 Systems of Linear Equations

We start with a theorem of elementary linear algebra.

Theorem 1. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Then the following conditions are equivalent:*
(1) for any $y \in \mathbb{F}^m$ there exists at most one vector $x \in \mathbb{F}^n$ such that $Ax = y$,
(2) $\exists\, y \in \mathbb{F}^m : \#\{x \in \mathbb{F}^n : Ax = y\} = 1$*,*
(3) $\operatorname{rank}(A) = n$*,*
(4) $\operatorname{Ker}(A) = \{\mathbf{0}\}$*.*

In [1], Donoho and Elad provided a generalization of the above result.

Theorem 2 (Donoho and Elad). *Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $k \in \mathbb{N}\cup\{0\}$. Then the following conditions are equivalent:*
(5) for any $y \in \mathbb{F}^m$ there exists at most one vector $x \in \mathbb{F}^n$ such that $Ax = y$ and $\|x\|_0 \leq k$,
(6) $\Sigma_{2k} \cap \operatorname{Ker}(A) = \{\mathbf{0}\}$*,*
(7) $\operatorname{spark}(A) > 2k$*.*

Proof. Equivalence *(6)* $\Leftrightarrow$ *(7)* is obvious. Notice that the Hamming weight of a vector $x \in \mathbb{F}^n$ is not greater than $2k$ if and only if $x = u - v$ for some $u, v \in \Sigma_k$. Consequently,

$$\neg\textit{(5)} \Leftrightarrow (\exists\, u, v \in \Sigma_k : u \neq v,\ Au = Av) \Leftrightarrow$$

$$\Leftrightarrow (\exists\, x \in \Sigma_{2k} \setminus \{\mathbf{0}\} : Ax = \mathbf{0}) \Leftrightarrow \neg\textit{(6)}.$$

□

We included the simple and well known proof of the Donoho-Elad theorem for the sake of completness. If $k = n$, then condition *(5)* becomes *(1)* of Theorem 1, condition *(6)* becomes *(4)*, and condition *(7)* becomes *(3)*. The example below shows that no counterpart of condition *(2)* can be added to Theorem 2.

Example 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is easy to see that $\{x \in \mathbb{F}^3 : Ax = [1,0]^{\mathrm{T}},\ \|x\|_0 \leq 1\} = \{[1,0,0]^{\mathrm{T}}\}$. However, $A[0,1,0]^{\mathrm{T}} = A[0,0,1]^{\mathrm{T}}$.

3 More about Spark Varieties

Let $r \in \{0,\dots,\min\{m,n\}\}$. Recall that the *generic determinantal variety*

$$\mathcal{H}^r_{m\times n}(\mathbb{F}) = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{rank}(A) \leq r\}$$

is an algebraic subset of $\mathcal{M}_{m\times n}(\mathbb{F})$. (Moreover, $\Lambda(\mathcal{H}^r_{m\times n}(\mathbb{F})) = r\max\{m,n\}$, and if $\mathbb{F}$ is an algebraically closed field, then $\mathcal{H}^r_{m\times n}(\mathbb{F})$ is irreducible of dimension $r(m+n-r)$). Consequently, $\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{rank}(A) > r\}$ is Zariski open in $\mathcal{M}_{m\times n}(\mathbb{F})$. Since $\min\{m,n\} = \max_{A\in\mathcal{M}_{m\times n}(\mathbb{F})} \operatorname{rank}(A)$, so is the set $\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{rank}(A) = \min\{m,n\}\}$. These facts and Theorem 2 yield a motivation for paying attention to the *spark varieties*

$$\mathcal{S}^k_{m\times n}(\mathbb{F}) = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{spark}(A) \leq k\},$$

where k is any positive integer. Notice that $\mathcal{S}^k_{m\times n}(\mathbb{F})$ coincides with the totality of matrices in $\mathcal{M}_{m\times n}(\mathbb{F})$ which have a $\min\{k,n\}$-element set of linearly dependent columns. The following simple properties have been proved in [4].

Theorem 3. *(i) Every spark variety $\mathcal{S}^k_{m\times n}(\mathbb{F})$ is an algebraic subset of the space $\mathcal{M}_{m\times n}(\mathbb{F})$.*
(ii) If $\min\{k,n\} > m$, then $\mathcal{S}^k_{m\times n}(\mathbb{F}) = \mathcal{M}_{m\times n}(\mathbb{F})$.
(iii) If $\min\{k,m\} \geq n$, then $\mathcal{S}^k_{m\times n}(\mathbb{F}) = \mathcal{H}^{n-1}_{m\times n}(\mathbb{F})$.
(iv) If $\mathbb{F}$ is infinite and $\min\{k,n\} \leq m$, then $\Lambda(\mathcal{S}^k_{m\times n}(\mathbb{F})) = m(n-1)$.
(v) If $\mathbb{F}$ is algebraically closed, $k \leq m$ and $k < n$, then $\mathcal{S}^k_{m\times n}(\mathbb{F})$ is reducible and has pure dimension $m(n-1)+k-1$.

Corollary 1. *Let $k \in \mathbb{N}\cup\{0\}$ and $\sigma = \max_{A\in\mathcal{M}_{m\times n}(\mathbb{F})} \operatorname{spark}(A)$. Then*
(i) $\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{spark}(A) > k\}$ is Zariski open in $\mathcal{M}_{m\times n}(\mathbb{F})$,
(ii) $\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{spark}(A) = \sigma\}$ is Zariski open in $\mathcal{M}_{m\times n}(\mathbb{F})$,
(iii) $\sigma = \begin{cases} +\infty, & \text{if } m \geq n, \\ m+1, & \text{if } m < n \text{ and } \mathbb{F} \text{ is infinite.} \end{cases}$
Moreover, if the field $\mathbb{F}$ is infinite, then

$$\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{spark}(A) > k\} \neq \emptyset \Leftrightarrow (m \geq n \vee k \leq m < n).$$

Proof. Property *(i)* is obvious. Notice that $\sigma \geq 2$. Property *(ii)* follows from the fact that

$$\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{spark}(A) = \sigma\} = \begin{cases} \mathcal{M}_{m\times n}(\mathbb{F}) \setminus \mathcal{S}^{\sigma-1}_{m\times n}(\mathbb{F}), \text{ if } \sigma \neq +\infty, \\ \mathcal{M}_{m\times n}(\mathbb{F}) \setminus \mathcal{S}^{n}_{m\times n}(\mathbb{F}), \text{ if } \sigma = +\infty. \end{cases}$$

If $m \geq n$, then $\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \mathrm{spark}(A) = +\infty\} = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \mathrm{rank}(A) = n\} \neq \emptyset$. Suppose, therefore, that $m < n$ and $\mathbb{F}$ is infinite. Let $\overline{\mathbb{F}}$ be the algebraic closure of the field $\mathbb{F}$. By Proposition 1 *(iv)*,

$$\mathrm{spark}(A) \leq \mathrm{rank}(A) + 1 \leq m + 1$$

for all $A \in \mathcal{M}_{m\times n}(\overline{\mathbb{F}})$. Since $\dim \mathcal{S}^m_{m\times n}(\overline{\mathbb{F}}) = m(n-1) + m - 1 = mn - 1$, the set

$$\{A \in \mathcal{M}_{m\times n}(\overline{\mathbb{F}}) : \mathrm{spark}(A) = m+1\} = \mathcal{M}_{m\times n}(\overline{\mathbb{F}}) \setminus \mathcal{S}^m_{m\times n}(\overline{\mathbb{F}})$$

is nonempty and Zariski open in $\mathcal{M}_{m\times n}(\overline{\mathbb{F}})$. The infiniteness of $\mathbb{F}$ yields that $\mathcal{M}_{m\times n}(\mathbb{F})$ is Zariski dense in $\mathcal{M}_{m\times n}(\overline{\mathbb{F}})$. Consequently, there exists a matrix $A_0 \in \mathcal{M}_{m\times n}(\mathbb{F})$ such that $\mathrm{spark}(A_0) = m + 1$, and hence $\sigma = m + 1$ (see Proposition 1 *(vi)*).

The "moreover" part follows directly from *(iii)*. □

In the proof of property *(iii)*, the existence of a matrix in $\mathcal{M}_{m\times n}(\mathbb{F})$ whose spark is equal to $m + 1$ can also be obtained by induction on n.

If $\mathbb{F}$ is a finite field, then $\max_{A\in\mathcal{M}_{m\times n}(\mathbb{F})} \mathrm{spark}(A)$ can be less than $m + 1$.

Example 2. Suppose that $\mathbb{F}$ is finite and $n \geq (\#\mathbb{F})^m$. Since the vector space $\mathbb{F}^m$ has exactly $(\#\mathbb{F})^m - 1$ elements different from $\mathbf{0}$, every matrix in $\mathcal{M}_{m\times n}(\mathbb{F})$ which has no zero column, must have two identical columns. Hence

$$\max_{A\in\mathcal{M}_{m\times n}(\mathbb{F})} \mathrm{spark}(A) = 2.$$

4 Some Algebraic Properties of the Spark

We begin with a very simple remark.

Proposition 2. *Let $\ell, s \in \mathbb{N}$, $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, $B \in \mathcal{M}_{n\times \ell}(\mathbb{F})$, and $\{j_1, \ldots, j_s\}$ be an s-element subset of $\{1, \ldots, n\}$. Define $A' \in \mathcal{M}_{m\times s}(\mathbb{F})$ to be the matrix consisting of the columns of A with indices $j_1, \ldots, j_s$. Then*
(i) $\mathrm{spark}(AB) \leq \mathrm{spark}(B)$,
(ii) $\mathrm{spark}(A') \geq \mathrm{spark}(A)$.

Proof. Since $\mathrm{Ker}(B) \subseteq \mathrm{Ker}(AB)$, inequality *(i)* follows from Proposition 1 *(v)*. Inequality *(ii)* is an immediate consequence of the definition of the spark. □

Let us recall that if $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $B \in \mathcal{M}_{n\times \ell}(\mathbb{F})$, then $\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$.

Example 3. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(over an arbitrary field), and observe that $\mathrm{spark}(A) = 2$ while $\mathrm{spark}(AB) = 3 = \mathrm{spark}(B)$.

Let $\ell \in \{1, \ldots, n\}$ and let $j_1, \ldots, j_\ell \in \{1, \ldots, n\}$ be such that $j_1 < \ldots < j_\ell$. We denote by $B^{(j_1,\ldots,j_\ell)}$ the principal $\ell \times \ell$ minor of a matrix $B = [\beta_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{F})$ defined by $j_1, \ldots, j_\ell$ (i.e., $B^{(j_1,\ldots,j_\ell)}$ is the determinant of the matrix $\left[\beta_{j_p j_q}\right]_{p,q=1,\ldots,\ell}$).

In the subsequent theorem we will work over $\mathbb{F} = \mathbb{C}$, the field of complex numbers. We denote the Hermitian conjugate of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ by A^*. For a positive integer k and vectors $x^1, \ldots, x^k \in \mathbb{C}^m$ we define $\mathrm{Gr}(x^1, \ldots, x^k)$ to be the determinant of the matrix $\left[(x^i)^* x^j\right]_{i,j=1,\ldots,k}$ (i.e., $\mathrm{Gr}(x^1, \ldots, x^k)$ is the Gram determinant of $x^1, \ldots, x^k$).

Theorem 4. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and let μ be the infimum of the set of all positive integers ℓ such that the matrix A^*A has a principal $\ell \times \ell$ minor equal to* 0. *Then $\mu = \mathrm{spark}(A) = \mathrm{spark}(A^*A)$.*

Proof. We define $B = A^*A$. Moreover, let $C_1, \ldots, C_n$ be the columns of A. Notice that

$$\forall\, \ell \in \{1, \ldots, n\}\, \forall\, j_1, \ldots, j_\ell \in \{1, \ldots, n\} :\, j_1 < \ldots < j_\ell \;\Rightarrow$$

$$\Rightarrow\; B^{(j_1,\ldots,j_\ell)} = \mathrm{Gr}(C_{j_1}, \ldots, C_{j_\ell}).$$

If ℓ is a positive integer and $j_1, \ldots, j_\ell \in \{1, \ldots, n\}$ are such that $j_1 < \ldots < j_\ell$, then

$$\mathrm{Gr}(C_{j_1}, \ldots, C_{j_\ell}) = 0 \iff C_{j_1}, \ldots, C_{j_\ell} \text{ are linearly dependent.}$$

The equality $\mathrm{spark}(A) = \mu$ follows therefore directly from the definition of the spark.

Now, Proposition 2 yields $\mathrm{spark}(A) \geq \mathrm{spark}(B)$. Suppose that

$$s := \mathrm{spark}(B) \neq +\infty.$$

Let $D_{p_1}, \ldots, D_{p_s}$, where $p_1, \ldots, p_s \in \{1, \ldots, n\}$ and $p_1 < \ldots < p_s$, be linearly dependent columns of the matrix B. Then every $s \times s$ minor of B consisting of elements of $D_{p_1}, \ldots, D_{p_s}$ is equal to 0. In particular, $B^{(p_1,\ldots,p_s)} = 0$. Since $B^{(p_1,\ldots,p_s)} = \mathrm{Gr}(C_{p_1}, \ldots, C_{p_s})$, it follows that $C_{p_1}, \ldots, C_{p_s}$ are linearly dependent, and hence $\mathrm{spark}(A) \leq s$. The proof is complete. ☐

Of course, the above proof is fully analogous to the very well known proof of the fact that $\mathrm{rank}(A) = \mathrm{rank}(A^*A) = \mathrm{rank}(AA^*)$. It is also clear that $\mathrm{spark}(AA^*) = \mathrm{spark}(A^*)$ can be different from $\mathrm{spark}(A)$.

Let us return to an arbitrary field $\mathbb{F}$. We will conclude the note by a theorem proved in [5], which describes other multiplicative properties of the spark. Recall that $\mathrm{rank}(UAV) = \mathrm{rank}(A)$ for any matrices $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, $U \in \mathcal{GL}_m(\mathbb{F})$ and $V \in \mathcal{GL}_n(\mathbb{F})$.

Theorem 5. *For a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ the following hold true:*

(i) $\forall\, U \in \mathcal{GL}_m(\mathbb{F}) :\ \mathrm{spark}(UA) = \mathrm{spark}(A)$,

(ii) $\mathrm{spark}(AD) = \mathrm{spark}(A)$ *for every diagonal matrix* $D \in \mathcal{GL}_n(\mathbb{F})$,

(iii) $\mathrm{spark}(AP) = \mathrm{spark}(A)$ *for every permutation matrix* $P \in \mathcal{M}_{n\times n}(\mathbb{F})$,

(iv) if $\mathrm{rank}(A) = n-1$, *then* $\forall\, s \in \{1,\dots,n\}\, \exists\, V \in \mathcal{GL}_n(\mathbb{F}) :\ \mathrm{spark}(AV) = s$,

(v) if A has no zero column, then $\exists\, V \in \mathcal{GL}_n(\mathbb{F}) :$
$\mathrm{spark}(AV) = \mathrm{spark}(A) - 1$.

Moreover, if the field $\mathbb{F}$ *has at least* $n+1$ *elements and* $\mathrm{rank}(A) = n-2$, *then*

$$\forall\, s \in \{1,\dots,n-1\}\, \exists\, V \in \mathcal{GL}_n(\mathbb{F}) :\ \mathrm{spark}(AV) = s.$$

References

1. D. L. Donoho and M. Elad, Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization, *Proc. Natl. Acad. Sci. USA* **100**, No. 5: 2197–2202 (2003).
2. R. Hartshorne, *Algebraic Geometry*, Springer, 1983.
3. G. Kutyniok, Theory and Applications of Compressed Sensing, arXiv:1203.3815 (August 2012).
4. M. Skrzyński, A note on spark varieties, *Acta Comment. Univ. Tartu. Math.*, to appear.
5. M. Skrzyński, Remarks on the spark of a matrix and the Null Space Property, submitted.

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On Transcendence of Certain Powers of e

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Abstract. In this paper we prove a particular case of the Hermite-Lindemann Transcendence Theorem: if α is a non-zero algebraic number, then e^α is transcendental. We also give some applications of that theorem.

1 Introduction

Proofs of transcendence of numbers e and π obtained in the nineteenth century ([5, 8]) were an inspiration for further intensive research which has been continued up to the present day (see e.g. [1, 3, 6, 9–12]). The following theorem (now called the Hermite-Lindemann Transcendence Theorem ([2]), proved by Lindemann in [8], was the first important fact which gave transcendence of e as well as π:

Theorem 1. *Any finite expression $A_1e^{\alpha_1}+A_2e^{\alpha_2}+\ldots+A_ie^{\alpha_i}+\ldots$, in which the coefficients A_i are non-zero algebraic numbers and the exponents α_i are distinct algebraic numbers, is never equal to zero.*

Theorem 1 is a generalization of the main result presented by Hermite in [5], where he assumed the coefficients (A) and the exponents (α) are integers.

Because of the generality of Theorem 1, existing proofs of this result are difficult and therefore generally not well known.

In this paper, we study a particular case of Theorem 1 from which transcendence of e, and also π, follows immediately: if α is a non-zero algebraic number, then e^α is transcendental (the result itself is sometimes called the Hermite-Lindemann theorem). Many proofs of this theorem can be found in the literature (see for example [7, 9–11]); here we present one more using classical methods as in [5] but including partially new reasoning. Our proof is self-contained and elementary as much as possible. The only prerequisite to understand this proof is the knowledge of algebra and calculus taught at an undergraduate level, together with some mathematical maturity.

2 Preliminaries

In the following, we let $\mathbb{C}$ denote the set of complex numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ prime field with p elements.

Before presenting our results, let us recall that a complex number α is said to be **algebraic** if there is a non-zero polynomial with integer coefficients of which α is a root. A complex number α which is not algebraic is said to be **transcendental**.

Let us also recall some easy but useful facts about complex integrals.

(a) If a function $f(z)$ is integrable along a curve γ, then

$$|\int_\gamma f(z)dz| \leq ML(\gamma),$$

where $M = max_{z\in\gamma}|f(z)|$ and $L(\gamma)$ is the length of γ.

(b) In particular, if γ is a segment connecting z_1 and z_2 (for $z_1, z_2 \in \mathbb{C}$), then

$$|\int_{z_1}^{z_2} f(z)dz| \leq M|z_1 - z_2|.$$

(c) Let $f(z)$ be a continuous function on D. If z_1 and z_2 are any two points in D and $F'(z) = f(z)$ on D, then

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1).$$

Hence, in this case, one can use classical methods of integration, in particular integration by parts.

From the above we obtain the following

Lemma 1. *If the functions $h_1(z)$ and $h_2(z)$ are continuous on a segment $[z_1, z_2]$, then*

$$\lim_{n\to\infty} \frac{1}{n!} \left| \int_{z_1}^{z_2} h_1(z)(h_2(z))^n dz \right| = 0.$$

Proof. Let $max_{z\in[z_1,z_2]}|h_1(z)| = |h_1(M_1)|$ and $max_{z\in[z_1,z_2]}|h_2(z)| = |h_2(M_2)|$ for some $M_1, M_2 \in [z_1, z_2]$. Set $c_1 = |h_1(M_1)|$ and $c_2 = |h_2(M_2)|$. It is clear that for all $n \in \mathbb{N}$, $|h_1(z)(h_2(z))^n| \leqslant c_1 c_2^n$ on $[z_1, z_2]$. Hence from b) we obtain that $\frac{1}{n!} \left| \int_{z_1}^{z_2} h_1(z)(h_2(z))^n dz \right| \leqslant |z_1 - z_2| c_1 \frac{c_2^n}{n!}$. Since $\lim_{n\to\infty} \frac{c_2^n}{n!} = 0$, the result follows. □

Finally, we will need some facts about symmetric polynomials. Denote by $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ the ring of polynomials over $\mathbb{Z}$ in indeterminates $x_1, x_2, \ldots, x_n$. Obviously $(x - x_1)(x - x_2)\ldots(x - x_n) = x^n - \sigma_1 x^{n-1} + \ldots + (-1)^n \sigma_n$, where $\sigma_i \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ for all $i = 1, 2, \ldots, n$. Recall that the $\sigma_1, \sigma_2, \ldots, \sigma_n$ are called elementary symmetric polynomials in indeterminates $x_1, x_2, \ldots, x_n$.

We shall need the fundamental theorem of symmetric polynomials (cf. [7] p. 190-194) in the following simplified version.

Theorem 2. *If $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is a symmetric polynomial of degree N, then there exists a polynomial $g \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, of degree at most N, such that $f(x_1, x_2, \ldots, x_n) = g(\sigma_1, \sigma_2, \ldots, \sigma_n)$.*

3 The Main Result

Theorem 3. *If $z \neq 0$ is an algebraic number, then e^z is transcendental.*

Proof. Let $z \neq 0$ be an algebraic number and assume to the contrary that e^z is also algebraic. Then z is a root of a polynomial $P(x) = b_0 + b_1 x + \ldots + b_m x^m$ with integer coefficients. Moreover, $a_0 + a_1 e^z + a_2 e^{2z} + \ldots + a_r e^{rz} = 0$ for some integer numbers $a_0, a_1, a_2, \ldots, a_r$ and $a_0 \neq 0$. Additionally we can assume that $a_0 + a_1 + \ldots + a_r \neq 0$. Let $z_1 = z$, $z_2, \ldots, z_m$ be all the roots of $P(x)$. We can assume that $b_0 \neq 0$ and $b_m > 0$.

I. Consider the following product

$$\begin{aligned} T = (a_0 + a_1 e^{z_1} + a_2 e^{2z_1} + \ldots + a_r e^{rz_1}) \\ \cdot (a_0 + a_1 e^{z_2} + a_2 e^{2z_2} + \ldots + a_r e^{rz_2}) \\ \cdot (a_0 + a_1 e^{z_3} + a_2 e^{2z_3} + \ldots + a_r e^{rz_3}) \cdot \ldots \\ \cdot (a_0 + a_1 e^{z_m} + a_2 e^{2z_m} + \ldots + a_r e^{rz_m}). \end{aligned} \tag{1}$$

Note that $T = 0$ since the first factor equals zero. It is easy to see that T is a symmetric function of variables $z_1, z_2, \ldots, z_m$. Multiplying out, we can write T in the form

$$T = B_1 + B_2 e^{\beta_2} + B_3 e^{\beta_3} + \ldots + B_s e^{\beta_s}, \tag{2}$$

where $B_1 = a_0^m$, $B_2 = a_0^{m-1} a_1$, …, $B_s(x) = a_r^m$, and $\beta_2 = z_1$, …, $\beta_s = rz_1 + rz_2 + \ldots + rz_m$.

After cancellation of similar terms in (2) we can renumerate indexes of exponents β_i. Then T can be written in the form

$$T = A_1 + A_2 e^{\beta_2} + A_3 e^{\beta_3} + \ldots + A_k e^{\beta_k}. \tag{3}$$

Since T is a symmetric function of variables $z_1, z_2, \ldots, z_m$ and $B_1 \neq 0$ we can assume that $A_i \neq 0$ for all $i = 1, 2, \ldots, k$.

We can choose $0 \neq t \in \mathbb{Z}$ such that $\beta_i + t \neq 0$ for all $i = 2, 3, \ldots, k$. Multiplying (3) by e^t, we can rewrite $e^t T$ in the form

$$e^t T = \sum_{i=1}^{k} A_i e^{\alpha_i},$$

where $\alpha_1 = t$ and $\alpha_i = t + \beta_i$ for all $i = 2, 3, \ldots, k$.

In conclusion,

$$A_1 e^{\alpha_1} + A_2 e^{\alpha_2} + \ldots + A_k e^{\alpha_k} = 0 \tag{4}$$

and $\sum_{i=1}^{k} A_i e^{\alpha_i}$ is a symmetric function of variables $z_1, z_2, \dots, z_m$, $A_i \neq 0$, $(a_0 + a_1 + \ldots + a_r)^m = A_1 + A_2 + \ldots + A_k \neq 0$ and $\alpha_1 = t$, $\alpha_i \neq \alpha_j$ for $i \neq j$.

II. Now let

$$g(x) = \prod_{i=1}^{k} (x - \alpha_i).$$

Moreover, take $h(x) = (x - \alpha_1)^b g(x)$, where b corresponds to the number of β_i that equals zero in (2). Clearly, the coefficients of $h(x)$ are symmetric in $z_1, z_2, \dots, z_m$. Therefore, if $h(x) = h_0 + h_1 x + \ldots + h_v x^v$, where $v = deg(g(x)) + b$, then $h_i = s_i(z_1, z_2, \dots, z_m)$ for some symmetric polynomial $s_i \in \mathbb{Z}[x_1, x_2, \dots, x_m]$ $(i = 0, 1, 2, \dots v)$. From Theorem 2, we have $s_i = r_i(\sigma_1, \sigma_2, \dots, \sigma_m)$ for some $r_i \in \mathbb{Z}[x_1, x_2, \dots, x_m]$. Since each $\sigma_i(z_1, z_2, \dots, z_m)$, $i = 1, 2, \dots, m$, is one of the coefficients of the polynomial $\frac{P(x)}{b_m}$, it follows that $h_i \in \mathbb{Q}$. So, the coefficients of $h(x)$ and consequently of $g(x)$ are rational. Hence, one can assume that $g(x)$ has integer coefficients. Now we define a polynomial

$$f(x) = \frac{b_m^N (g(x))^n}{n!}, \tag{5}$$

where $n \in \mathbb{Z}$ and $N = n(deg(g(x)) - 1)$.

Next, for arbitrary $c \in \mathbb{C}$, let

$$I(c) = \int_0^c e^{c-x} f(x) dx. \tag{6}$$

Applying Lemma 1 to $h_1(x) = e^{c-x}$ and $h_2(x) = b_m^{deg(g(x))-1} g(x)$, $z_1 = 0$ and $z_2 = c$, we have that

$$|I(c)| \to 0 \ \text{ as } \ n \to \infty. \tag{7}$$

Using integration by parts, one can show that

$$I(c) = e^c G(0) - G(c), \tag{8}$$

where $G(x) = f^{(0)}(x) + f^{(1)}(x) + \ldots + f^{(S)}(x)$ and $S = deg(f(x))$.

Therefore from (7), (8) and next from (4), for large enough n, we obtain

$$\begin{aligned}
&1 > |A_1 I(\alpha_1)| + |A_2 I(\alpha_2)| + \ldots + |A_k I(\alpha_k)| \\
&= |A_1(G(0)e^{\alpha_1} - G(\alpha_1))| + |A_2(G(0)e^{\alpha_2} - G(\alpha_2))| + \ldots + |A_k(G(0)e^{\alpha_k} - G(\alpha_k))| \\
&\geq |G(0)(\sum_{i=1}^{k} A_i e^{\alpha_i}) - \sum_{i=1}^{k} A_i G(\alpha_i))| \\
&= |\sum_{i=1}^{k} A_i G(\alpha_i)|.
\end{aligned}$$

So

$$1 > |\sum_{i=1}^{k} A_i G(\alpha_i)|. \tag{9}$$

III. Now we will show that $\sum_{i=1}^{k} A_i G(\alpha_i)$ is a non-zero integer for infinitely many n. Let p be a prime number and $n = p^w - 1$ for $w \in \mathbb{N}$.

Since $G(x)$ is expressed in terms of $f^{(i)}$, we need the following properties of $f^{(i)}$ (they all follow directly from the definition (5) of f).

(i) if $d < n$, then $f^{(d)}(\alpha) = 0$ for every root α of $g(x)$,
(ii) if $d = n$, then $f^{(d)}(\alpha_i) = b_m^N g_i(\alpha_i)^n$, where $g_i(x) = \frac{g(x)}{x-\alpha_i}$ for all $i = 1, 2, \ldots, k$
(iii) if $d > n$, then $f^{(d)}(x)$ is a polynomial with integer coefficients divisible by pb_m^N.

Let us look more closely at the sum $\sum_{i=1}^{k} A_i f^{(d)}(\alpha_i)$.

Of course, $\sum_{i=1}^{k} A_i f^{(d)}(\alpha_i)$ is a symmetric polynomial of degree at most N in variables $z_1, z_2, \ldots, z_m$ for $d \geqslant n$. Applying (ii) and (iii) we obtain that $\sum_{i=1}^{k} A_i f^{(d)}(\alpha_i) = z(d) b_m^N w_d(z_1, z_2, \ldots, z_m)$ for some symmetric polynomial $w_d \in \mathbb{Z}[x_1, x_2, \ldots, x_m]$, where $z(d) = 1$ for $d = n$ and $z(d) = p$ for $d > n$. From Theorem 2 we have $w_d = e_d(\sigma_1, \sigma_2, \ldots, \sigma_m)$ for some $e_d \in \mathbb{Z}[x_1, x_2, \ldots, x_m]$ of degree at most N. Since each $\sigma_i(z_1, z_2, \ldots, z_m)$, $i = 1, 2, \ldots, m$, is one of the coefficients of the polynomial $\frac{P(x)}{b_m}$, we can see easily that $\sum_{i=1}^{k} A_i f^{(d)}(\alpha_i)$ is an integer divisible by $z(d)$.

It is clear that $g_i(\alpha_i) \neq 0$ is an algebraic number for all $i = 1, 2, \ldots, k$. Thus there exists $a \in \mathbb{C}$ such that $\mathbb{Q}(g_1(\alpha_1), g_2(\alpha_2), \ldots, g_k(\alpha_k)) = \mathbb{Q}(a)$. We can assume (multiplying a if necessary by some integer) that a minimal polynomial $k(x) = x^s + z_{s-1}x^{s-1} + \ldots + z_1 x + z_0$ for a over $\mathbb{Q}$ is monic and $k(x) \in \mathbb{Z}[x]$. Consequently, there exists $b \in \mathbb{Z}$, such that $bg_i(\alpha_i) \in \mathbb{Z}[a]$ for all i, where $\mathbb{Z}[a] \cong \mathbb{Z}[x]/k(x)\mathbb{Z}[x]$. Let M_p be a maximal ideal of $\mathbb{Z}[a]$ such that $p\mathbb{Z}[a] \subseteq M_p$, where p is a prime number. Clearly $\mathbb{Z}[a]/M_p$ is a finite field with p^w elements for some positive integer w. Let $\overline{g}\colon \mathbb{Z}[a] \to \mathbb{Z}[a]/M_p$ be a homomorphism defined by $x \mapsto \overline{x} = (x + M_p)/M_p$ for $x \in \mathbb{Z}[a]$. It is not difficult to show that for any infinite set T of primes, $\bigcap_{p \in T} M_p = 0$. Therefore for infinitely many prime numbers p, $\overline{bg_i(\alpha_i)} \neq 0$ for all i. But $\overline{b_m^N (bg_i(\alpha_i))^{p^w - 1}} = \overline{b^{p^w-1} f^{(p^w-1)}(\alpha_i)}$ from (ii). Hence for p large enough, $\overline{A_i b^{p^w-1} f^{(p^w-1)}(\alpha_i)} = \overline{A_i b^{p^w-1}} \cdot \overline{f^{(p^w-1)}(\alpha_i)} = \overline{A_i b_m^N}$ for all i. So $\overline{\sum_{i=1}^{k} A_i f^{(p^w-1)}(\alpha_i)} = \overline{(A_1 + A_2 + \ldots + A_k) b_m^N}$. Clearly if $q \in \mathbb{Z}$ then $\overline{q} \neq 0$ if and only if $p \nmid q$. So if $p \nmid (A_1 + A_2 + \ldots + A_k) b_m$ and $n = p^w - 1$, then $p \nmid \sum_{i=1}^{k} A_i f^{(n)}(\alpha_i)$. On the other hand we obtained that $\sum_{j=n+1}^{S} \sum_{i=1}^{k} A_i f^{(j)}(\alpha_i)$ is an integer divisible by p.

Since

$$\sum_{i=1}^{k} A_i G(\alpha_i) = \sum_{i=1}^{k} A_i f^{(n)}(\alpha_i) + \sum_{j=n+1}^{S} \sum_{i=1}^{k} A_i f^{(j)}(\alpha_i),$$

then we have that $\sum_{i=1}^{k} A_i G(\alpha_i)$ is a non-zero integer.

Hence, for p large enough and such that $n = p^{w(p)} - 1$ satisfies (9), we obtain from the above that

$$1 > |\sum_{i=1}^{k} A_i G(\alpha_i)| = |q|$$

for some $0 \neq q \in \mathbb{Z}$, so we are done. We get the contradiction and the theorem is proven. □

Corollary 1. *The following numbers are transcendental:*

(1) π,
(2) $\sin(\alpha)$, $\cos(\alpha)$ *for every non-zero algebraic number* α,
(3) $\log(\alpha)$ *for every algebraic number* $\alpha \neq 0, 1$.

Proof. (1) Clearly, $e^{i\pi} = -1$ is an algebraic number. Thus $i\pi$ and so π are transcendental.

(2) Assume to the contrary that $\sin(\alpha) = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$ is algebraic. Then it is easy to show that $e^{i\alpha}$ is algebraic over $\mathbb{Q}(i)$. Hence $e^{i\alpha}$ is algebraic. It follows that $i\alpha$ is transcendental, a contradiction. Analogously for $\cos(\alpha)$.

(3) Since $e^{\log(\alpha)} = \alpha$ is algebraic, $\log(\alpha)$ is transcendental. □

Using (6) and (8) from the proof of Theorem 3, and Lemma 1, we additionally can get interesting approximation of e^c for any positive real number c by considering different polynomials $f(x)$.

Example 1. Let $c \in \mathbb{R}$ and $c > 0$. Define $f(x) = \frac{[x(c-x)]^n}{n!}$ for some natural number n. Clearly, $f(x) = f(c-x)$. Hence $f^{(s)}(0) = (-1)^s f^{(s)}(c)$ for $s \in \mathbb{N}$.

Note that if $0 \leq s \leq n$ and $s \in \mathbb{Z}$ then

$$f^{(n+s)}(x) = \frac{1}{n!}\sum_{i=s}^{n}(-1)^i c^{n-i}\binom{n}{i}\frac{(n+i)!}{(i-s)!}x^{i-s}.$$

Therefore

(i) $f^{(n+s)}(0) = \frac{1}{n!}c^{n-i}\binom{n}{s}(n+s)!(-1)^s$,
(ii) $f^{(n+s)}(1) = \frac{1}{n!}c^{n-i}\binom{n}{s}(n+s)!(-1)^n$.

Now, as in the proof of Theorem 3, consider

$$I(c) = \int_0^c e^{c-x} f(x)dx = e^c G(0) - G(c),$$

where $G(x) = f^{(n)}(x) + f^{(n+1)}(x) + \ldots + f^{(2n)}(x)$. Applying (i) and (ii), we get

1) $G(0) = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i c^{n-i}\binom{n}{i}(n+i)!$,
2) $G(c) = \frac{1}{n!}(-1)^n\sum_{i=0}^{n} c^{n-i}\binom{n}{i}(n+i)!$.

Since $I(c) \to 0$ as $n \to \infty$,

$$e^c \approx \left|\frac{G(c)}{G(0)}\right| = \frac{\sum_{i=0}^{n} c^{n-i}\binom{n}{i}(n+i)!}{|\sum_{i=0}^{n}(-1)^i c^{n-i}\binom{n}{i}(n+i)!|} \tag{10}$$

is correct to at least t decimal places, where t is a maximal natural number satisfying $10^t \leq |G(0)|$. As n increases, the accuracy of (10) also increases. Moreover, this accuracy increases faster than t.

The particular cases of (10) for $c = 1$ are given in the following table.

Table 1.

$f(x) = \frac{[x(1-x)]^n}{n!}$					
$n =$	2	3	4	5	6
$e \approx$	$\frac{19}{7}$	$\frac{193}{71}$	$\frac{2721}{1001}$	$\frac{49171}{18089}$	$\frac{1084483}{398959}$
e correct to	2 dec. pl.	3 dec. pl.	6 dec. pl.	9 dec. pl.	11 dec. pl.

Example 2. Now consider

$$f(x) = \frac{(x(x-1)(x-2)\ldots(x-m))^n}{n!},$$

where $m, n \in \mathbb{N}$.

In this case we cannot express the $G(i)$ $(i = 1, 2, \ldots, m)$ by a closed formula even for $m = 2$. But, using a computer algebra system, we can easily compute the approximations of powers of e:

$$e \approx \frac{G(1)}{G(0)}, \quad e^2 \approx \frac{G(2)}{G(0)}, \quad \ldots, \quad e^m \approx \frac{G(m)}{G(0)}.$$

In the following table we list the approximations of e and e^2:

Table 2.

$f(x) = \dfrac{[x(x-1)(x-2)]^n}{n!}$				
$n =$	2	3	4	5
$e \approx$	$\dfrac{337}{124}$	$\dfrac{58019}{21344}$	$\dfrac{19363561}{7123456}$	$\dfrac{10641123569}{3914650592}$
e correct to	2 dec. pl.	6 dec. pl.	9 dec. pl.	13 dec. pl.
$e^2 \approx$	$\dfrac{229}{31}$	$\dfrac{9857}{1334}$	$\dfrac{1644863}{222608}$	$\dfrac{903924151}{122332831}$
e^2 correct to	2 dec. pl.	5 dec. pl.	9 dec. pl.	13 dec. pl.

Hermite in [5] obtains the following approximations

$$e \approx \frac{58291}{21444}, \qquad e^2 \approx \frac{158452}{21444}$$

correct to five and three decimal places, respectively. They are similar to the presented above for $n = 3$:

$$e \approx \frac{58019}{21344}, \qquad e^2 \approx \frac{9857}{1334} = \frac{157712}{21344}.$$

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References

1. F. Beukers, J.P. Bezivin, P. Robba, An alternative proof of the Lindemann-Weierstrass theorem, *American Mathematical Monthly*, **97** (1990), 193–197.
2. H. Dörrie, *100 Great Problems of Elementary Mathematics: Their History and Solutions*, Dover Publications, Inc. New York, 1965.
3. A.O. Gelfond, *Transcendental and Algebraic Numbers*, Dover Publications, Inc., 1960.

4. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, Oxford University Press, 1983.
5. C. Hermite, Sur la fonction exponentielle, *Comptes Rendus Acad. Sci. Paris* **77** (1873), 18–24.
6. D. Hilbert, Über die Transcendenz der Zahlen e und π, *Mathematische Annalen,* **43**(1893) 216–219.
7. S. Lang, *Algebra*, Revised Third Edition, Springer, 2005.
8. C.L.F. Lindemann, Über die Zahl π, *Mathematische Annalen,* **20** (1882) 213–225.
9. J. Pila, Geometric and Arithmetic Postulation of the Exponential Function, *The Journal of Australian Mathematical Society,* **54A** (1993), 111–127.
10. R.M. Redheffer and R. Steinberg, Analytic proof of the Lindemann theorem, *Pacific Journal of Mathematics,* **2** (1952), 231–242.
11. M. Waldschmidt, Linear independence measures for logarithms of algebraic numbers, Diophantine approximation (Cetraro, 2000), *Lecture Notes in Mathematics*, **1819**, Springer, Berlin, (2003) 250—344. Preprint version available at `http://www.math.jussieu.fr/~miw/articles/cetraro.html`.
12. K. Weierstrass, Zu Lindemann's Adhandlung, "Über die Ludolph'sche Zahl", *Math. Werke II* (1895), 341–362.

Part II

Geometry

Recent Results in Pure and Applied Mathematics, Podlasie 2014
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Projective Realizability of Veronese Spaces

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Abstract. We prove that a generalized Veronesian $\mathbf{V}_k(\mathfrak{M})$ cannot be realized in any Desarguesian projective space if only $k \geq 3$ and partial linear space $\mathfrak{M}$ contains a line on at least 4 points or $k > 3$ and partial linear space $\mathfrak{M}$ contains a line on at least 3 points. We have obtained this result using methods of the theory of combinatorial Veronesians. As a consequence of this fact we obtain that there is no Desarguesian projective space containing $\mathbf{V}_k(\mathfrak{P})$ where $\mathfrak{P}$ is a projective space and $k \geq 3$. We also solve the problem of realizability of $\mathbf{V}_2(PG(2,2))$ in $PG(n,2)$ and $\mathbf{V}_2(\mathfrak{V})$ in $PG(5,2)$ where $\mathfrak{V}$ is the Veblen (Pasch) configuration.

Keywords: Veronese space, combinatorial Veronese space, projective realization/embedding, Fano configuration/Fano space

1 Introduction

Originally, Veronese spaces appeared as varieties of some kind. After pioneer works of Tallini, Bichara, Melone, Mazzoca et al. (cf. [6], [3]) a Veronese space can be considered as a partial linear space that satisfies certain axioms; the models of this axiom system are structures of prisms in projective spaces. Therefore, the models constitute a world that can be imagined in a "anschauliche" geometry.

Definition of a structure of the prisms of hyperplanes of a projective space was generalized in [4] so as the points of the defined there *m-th generalized Veronese space over a partial linear space* $\mathfrak{M}$ (denoted by $\mathbf{V}_m(\mathfrak{M})$) are arbitrary m-multisets with elements in the pointset of $\mathfrak{M}$. The particular case where $\mathfrak{M}$ is a projective space was investigated in [1].

A natural question arises whether so generalized Veronese space can be embedded into a Desarguesian projective space.

A particular case of the construction of [4] where $\mathfrak{M}$ is a single k-element line X (in that case we write $\mathbf{V}_m(X) = \mathbf{V}_m(k)$) was intensively studied in [5].

In the particular case when $\mathfrak{M}$ is a single 3-element line L, $\mathbf{V}_3(L) = \mathbf{V}_3(3)$ is the known 10_3G configuration of Kantor (cf. [2]). It is known that $\mathbf{V}_3(3)$ can be realized on a real projective plane. But it was proved in [5] that this is an exceptional case: for $k > 3$ or $m > 3$ the structure $\mathbf{V}_m(k)$ cannot be embedded into a Desarguesian projective space.

From this we infer that for a huge class of partial linear spaces (even projectively realizable) their m-th Veronese structures cannot be realized in any Desarguesian projective space.

2 Definitions and problems

2.1 Veronese space

Let $\mathfrak{P}$ be a projective space (of dimension at least 3). Denote the class of the hyperplanes in $\mathfrak{P}$ by $\mathcal{H}$. Consider the following incidence structure called the Veronese space over $\mathfrak{P}$:

- points of the structure are the unordered pairs of hyperplanes (so called "double" hyperplanes are admitted);
- the points are grouped into blocks. The blocks may have one of the following two forms: one element of the pair is fixed, the second varies in a pencil of hyperplanes; another form: a family of double hyperplanes varying in a pencil of hyperplanes.

An axiomatic characterization of the Veronese spaces was given by Melone and Tallini in [3], see also [6].

Remark 1. Let $\mathcal{P}$ be the class of pencils of hyperplanes in $\mathfrak{P}$. The structure

$$\mathfrak{P}^\circ = \langle \mathcal{H}, \mathcal{P} \rangle$$

is a projective space.

2.2 Generalized Veronese space

Let k be a positive integer and X a set; we write $\wp_k(X)$ for the set of all k-subsets of X.

A multiset with repetitions (*a multiset*) of cardinality k with elements in the set X is a function $f\colon X \longrightarrow \mathbb{N}$ such that $|f| := \sum_{x\in X} f(x) = k < \infty$. We write $\mathfrak{y}_k(X)$ for the family of all such multisets. Clearly, if $f \in \mathfrak{y}_k(X)$ then $\mathrm{supp}(f) = \{x \in X\colon f(x) \neq 0\}$ is finite; in particular, if X is finite, we can identify f with the (formal) polynomial

$$f = \prod_{x\in\mathrm{supp}(f)} x^{f(x)} = \prod_{x\in X} x^{f(x)}$$

with variables in X.

A link between the naive usage of the term "multiset" and the formal one given here is explained by the formula $f = \{\underbrace{x_1,\dots,x_1}_{f(x_1)\text{ times}},\dots,\underbrace{x_t,\dots,x_t}_{f(x_t)\text{ times}},\dots\}$, where $\mathrm{supp}(f) = \{x_1,\dots,x_t,\dots\}$.

Remark 2. Let $|S| = n$. Then $|\mathfrak{y}_m(S)| = \binom{n+m-1}{m}$.

The (m-th) *Veronesian* over an incidence structure (block design) $\mathfrak{M} = \langle S, \mathcal{B}\rangle$ is the incidence structure

$$\mathbf{V}_m(\mathfrak{M}) := \langle \mathfrak{y}_m(S), \mathcal{B}^*\rangle,$$

where

$$\mathcal{B}^* = \left\{ eB^r \colon 1 \leq r \leq m,\ e \in \mathfrak{y}_{m-r}(S), B \in \mathcal{B} \right\}$$

and $eB^r = \{ex^r \colon x \in B\}$ (cf. [4]).

Fact 1. *The Veronese space over $\mathfrak{P}$ defined in Subsection 2.1 is exactly $\mathbf{V}_2(\mathfrak{P}^\circ)$.*

It is easily seen that

Fact 2. *If any two distinct blocks of a structure $\mathfrak{M}$ have at most λ common points then any two distinct blocks of $\mathbf{V}_m(\mathfrak{M})$ share at most λ common points.*

Corollary 1. *If $\mathfrak{M}$ is a partial linear space then $\mathbf{V}_m(\mathfrak{M})$ is also a partial linear space.*

It is easy to compute with the help of Rem. 2 the parameters of $\mathbf{V}_m(\mathfrak{M})$ where $\mathfrak{M}$ is a configuration.

Fact 3. *If $\mathfrak{M}$ is a $(v_r\, b_\kappa)$-configuration then $\mathbf{V}_m(\mathfrak{M})$ is a $\left(\binom{v+m-1}{m}_{mr}\, b \cdot \binom{v+m-1}{m-1}_\kappa\right)$-configuration.*

By an *embedding* of a block design $\langle S', \mathcal{B}' \rangle$ into a block design $\langle S, \mathcal{B} \rangle$ we mean an injective map which associates with the elements of S' elements of S and with the blocks in $\mathcal{B}'$ blocks in $\mathcal{B}$ which preserves in both directions the incidence (the membership relation). Frequently, it suffices to consider a respective injection defined for points only and require that for each block in $\mathcal{B}'$ its image under this injection is contained in a unique block in $\mathfrak{B}$ so as the induced map on blocks is an injection.

Let us fix an incidence structure $\mathfrak{M} = \langle S, \mathcal{B} \rangle$. Let us give fundamental facts concerning embeddings between Veronesians.

Fact 4. *The map $S \ni a \longmapsto a^1 \in \mathfrak{y}_1(S)$ is an isomorphism of $\mathfrak{M}$ onto $\mathbf{V}_1(\mathfrak{M})$.*

Fact 5. *Let $e \in \mathfrak{y}_k(S)$. The map $\mathfrak{y}_m(S) \ni f \longmapsto ef \in \mathfrak{y}_{m+k}(S)$ embeds $\mathbf{V}_m(\mathfrak{M})$ into $\mathbf{V}_{k+m}(\mathfrak{M})$.*

Fact 6. *Let k be a positive integer. The map $\mathfrak{y}_m(S) \ni f \longmapsto f^k \in \mathfrak{y}_{mk}(S)$ embeds $\mathbf{V}_m(\mathfrak{M})$ into $\mathbf{V}_{km}(\mathfrak{M})$.*

Corollary 2. *The maps $S \ni a \longmapsto a^m \in \mathfrak{y}_m(S)$ and, for each $e \in \mathfrak{y}_{m-1}(S)$, $S \ni a \longmapsto ea^1 \in \mathfrak{y}_m(S)$ are embeddings of $\mathfrak{M}$ into $\mathbf{V}_m(\mathfrak{M})$.*

Fact 7. *Let $\mathfrak{M}' = \langle S', \mathcal{B}' \rangle$ be a substructure of the incidence structure $\mathfrak{M}$. The natural inclusion $S' \subset S$ extends to the inclusion $\mathfrak{y}_m(S') \subset \mathfrak{y}_m(S)$, which, in turn, is an embedding of $\mathbf{V}_m(\mathfrak{M}')$ into $\mathbf{V}_m(\mathfrak{M})$.*

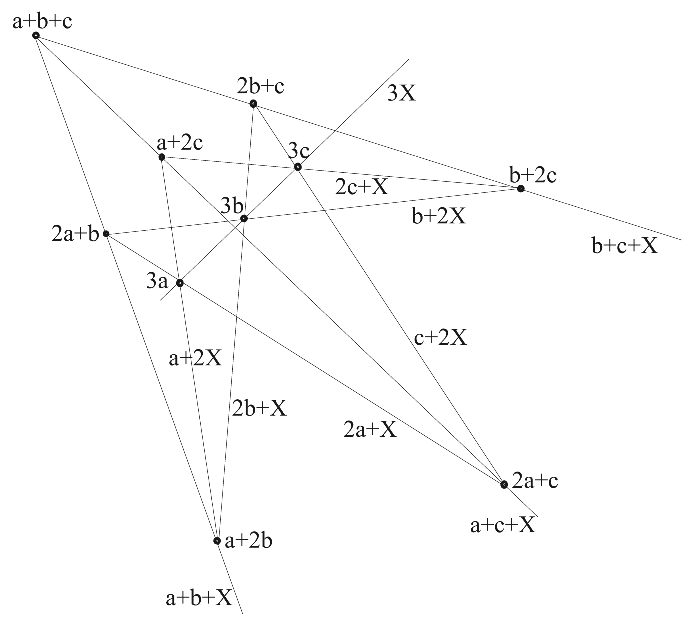

Fig. 1. The Kantor 10_3G-configuration

2.3 Combinatorial Veronesian

Combinatorial Veronesian is any structure $\mathbf{V}_m(\mathfrak{X})$, where $\mathfrak{X} = \langle X, \{X\}\rangle$ for a nonempty set X (cf. [5]). In short, we write $\mathbf{V}_m(X)$ instead of $\mathbf{V}_m(\langle X, \{X\}\rangle)$. We write also $\mathbf{V}_m(n) = \mathbf{V}_m(X)$, where $|X| = n$.

Remark 3. For a finite n the structure $\mathbf{V}_m(n)$ is a configuration of binomial type; precisely: it is a $\left(\binom{n+m-1}{m}_m \binom{n+m-1}{m-1}_n\right)$-configuration.

Example 1.

(1) $\mathbf{V}_m(2)$ coincides with the complete graph K_{m+1}.
(2) $\mathbf{V}_2(3)$ is the Veblen (Pasch) configuration.
(3) More generally, $\mathbf{V}_2(n)$ is the configuration of $n+1$ lines, each on n points, no three concurrent, but any two intersecting each other. It is dual to $\mathbf{V}_n(2)$.
(4) $\mathbf{V}_3(3)$ is the 10_3G Kantor configuration, see Figure 1 (comp. [5], [2]). ○

Remark 4. $\mathbf{V}_3(3)$ is self-dual. Moreover, $\mathbf{V}_m(n)$ and $\mathbf{V}_{m'}(n')$ are dual only in the three cases: $n = m = 3 = n' = m'$; $n = 2 = m'$, $n' = m$; and $m = 2 = n'$, $m' = n$.

2.4 Projective realizability

The question which we want to solve here is the following: for which structures $\mathfrak{M}$ and integers m we can embed $\mathbf{V}_m(\mathfrak{M})$ into a Desarguesian projective space.

An embedding of a partial linear space $\mathfrak{M}$ into a (Desarguesian!) projective space $\mathfrak{P}$ is called a *projective embedding* of $\mathfrak{M}$ into $\mathfrak{P}$ or a *projective realization* of $\mathfrak{M}$ in $\mathfrak{P}$. The structure $\mathfrak{M}$ has a projective realization means that there is a (Desarguesian!) projective space into which $\mathfrak{M}$ can be embedded.

Let $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ be a partial linear space. One thing is evident:

Fact 8. *$\mathfrak{M}$ is a substructure of $\mathbf{V}_m(\mathfrak{M})$ (cf. 2). Consequently, if $\mathbf{V}_m(\mathfrak{M})$ has a realization in a projective space $\mathfrak{P}$ then $\mathfrak{M}$ has a realization in $\mathfrak{P}$ as well.*

The following is known.

Fact 9. *$\mathbf{V}_3(3)$, $\mathbf{V}_2(n)$, and $\mathbf{V}_n(2)$ can be realized in real projective spaces.*

Fact 10 ([5, Prop. 6.5]). *The smallest projective space in which $\mathbf{V}_3(3)$ can be realized is $PG(2,7)$.*

Let us quote two results of [5] that are relevant here

Fact 11 ([5, Prop. 6.9]). *Let $k, m \geq 3$. If $k > 3$ or $m > 3$ then $\mathbf{V}_m(k)$ cannot be realized in any Desarguesian projective space (respective 'obstacle' configurations are presented in Figures 2 and 3).*

Fact 12 ([5, Prop. 6.4]). *$\mathbf{V}_3(3)$ cannot be realized in any Fano Desarguesian projective space.*

3 Results

It is evident that for each simple graph $\mathfrak{G}$ the structure $\mathbf{V}_k(\mathfrak{G})$ is also a simple graph. This yields immediately

Fact 13. *$\mathbf{V}_k(\mathfrak{G})$ has a projective realization for each simple graph $\mathfrak{G}$ and each positive integer k.*

Theorem 14. *Assume that one of the following holds:*

(i) either $k > 3$ and $\mathfrak{M}$ is a partial linear space with an at least 3-element line or
(ii) $k \geq 3$ and $\mathfrak{M}$ is a partial linear space with an at least 4-element line.

There is no Desarguesian projective space in which $\mathbf{V}_k(\mathfrak{M})$ can be embedded.

Proof. Suppose that $\mathbf{V}_k(\mathfrak{M})$ is contained in a Desarguesian projective space $\mathfrak{P}$. Let L be a line of $\mathfrak{M}$ with $|L| \geq 3$ in case (i) and with $|L| \geq 4$ in case (ii). Accordingly, we take $X \subset L$ such that $|X| = 3$ or $|X| = 4$ resp. Then $\mathbf{V}_k(\mathfrak{M})$ contains as a subspace the substructure $\mathbf{V}_k(L)$, and, in turn, $\mathbf{V}_k(L)$ contains $\mathbf{V}_k(X)$. Consequently, our embedding yields an embedding of $\mathbf{V}_k(X)$ into $\mathfrak{P}$, which contradicts Fact 11. $\square$

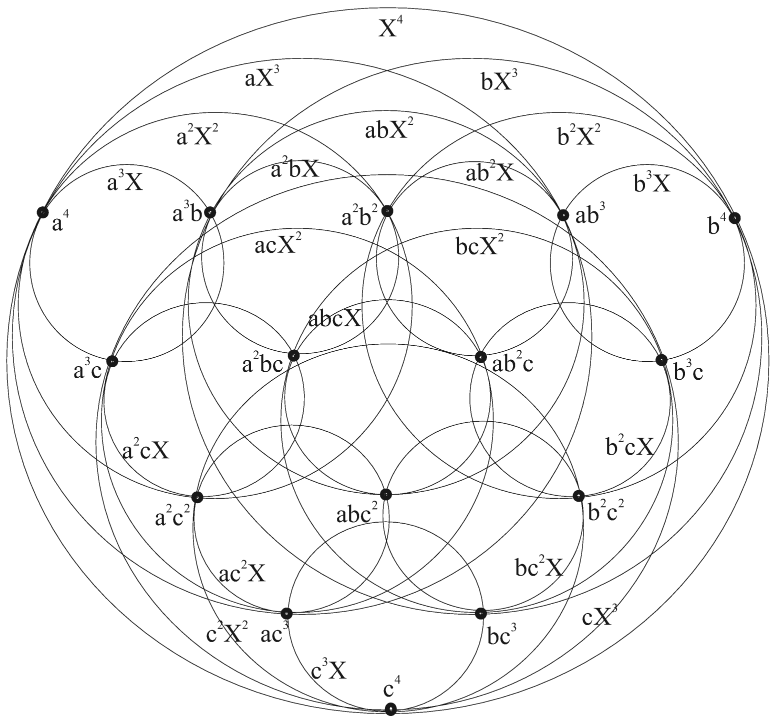

Fig. 2. The configuration $\mathbf{V}_4(3)$, it has no projective realization.

Theorem 15. *Let $\mathfrak{M}$ be a projective space and $k \geq 3$. There is no Desarguesian projective space in which $\mathbf{V}_k(\mathfrak{M})$ can be embedded.*

Proof. If $k > 3$ then the claim follows by Thm. 14(i). Suppose $k = 3$. If $\mathfrak{M}$ contains an at least 4-element line, the claim follows by Thm. 14(ii). However, if $\mathfrak{M}$ has only 3-element lines then it is a Fano space. Suppose $\mathbf{V}_3(\mathfrak{M})$ is embedded into a projective space $\mathfrak{P}$. Then $\mathfrak{P}$ contains a closed Fano subconfiguration and thus its characteristic is 2. In that case we arrive to a contradiction with Fact 12. □

Corollary 3. *If $k \geq 3$ and $\mathfrak{M}$ contains a projective space then there is no Desarguesian projective space in which $\mathbf{V}_k(\mathfrak{M})$ can be embedded.*

An illustrative application of the above is: a k-th Veronese space associated with a configuration which contains the Fano configuration has no projective realization.

However, there are also *2-th* Veronesians that can not be realized in some ways.

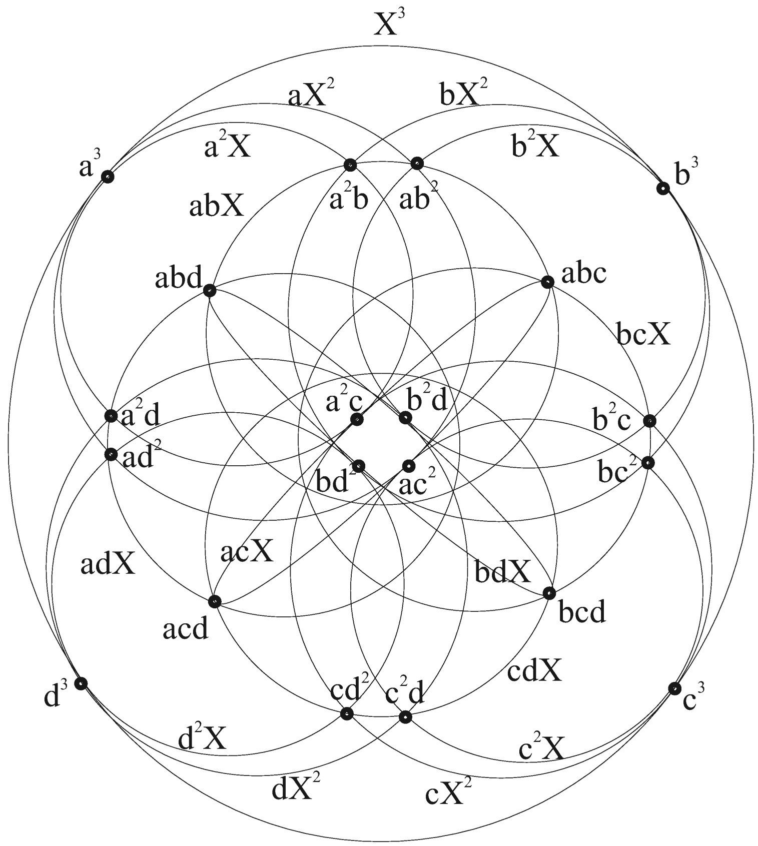

Fig. 3. The configuration $\mathbf{V}_3(4)$, it has no projective realization.

Theorem 16. *The Veronese space* $\mathbf{V}_2(\mathfrak{P}_0)$ *where* $\mathfrak{P}_0$ *is the Fano plane cannot be realized in any projective space* $PG(n, 2)$.

But, rather surprisingly, we have

Proposition 1. *Let* $\mathfrak{V}$ *be the Veblen configuration. The structure* $\mathbf{V}_2(\mathfrak{V})$ *can be realized in* $PG(5, 2)$.

Proof (simultaneously of Thm. 16 and of Prop. 1). We write $a \circledcirc b$ for the third point on the line $\overline{a, b}$ which joins points a, b in the (currently investigated) Steiner triple system.
Let us label the points of the Fano plane $PG(2, 2) = \langle S, \mathcal{L} \rangle$ so as points p_1, p_2, p_3 yield a triangle, $q_i = p_j \circledcirc p_k$ for $\neq (i, j, k)$, and $r = p_1 \circledcirc q_1$ ($= p_2 \circledcirc q_2 = p_3 \circledcirc q_3$). Suppose that $\mathfrak{M} = \mathbf{V}_2(PG(2, 2))$ is embedded into a projective space

$\mathfrak{P} = PG(n,2)$ for some integer n; clearly, $n \geq 4$. Consider the family $\mathcal{F} = \{S^2\} \cup \{xS \colon x \in S\}$; it consists of planes of $\mathfrak{P}$ and, simultaneously, of the (Fano) subplanes of $\mathfrak{M}$. Through each point of $\mathfrak{M}$ there pass two planes in $\mathcal{F}$ and any two planes in $\mathcal{F}$ share exactly one common point; consequently, no three planes in $\mathcal{F}$ have a common point. Let us consider one of such planes $\Pi_0 = S^2$ and let $\Pi_1 = p_1 S$ be any other subplane of $\mathfrak{M}$. Then the map $f_1 \colon x^2 \longmapsto p_1 x$ is a collineation of Π_0 onto Π_1, characterized by the condition

> the subplane of $\mathfrak{M}$ passing through x^2 and distinct from Π_0 crosses Π_1 in $p_1 x = f_1(x^2)$.

Recall that $a \circledcirc b = a + b$ holds in the $(n+1)$-dimensional vector space $\mathbb{V}$ over $GF(2)$ which represents $\mathfrak{P}$.

The vectors $e_1 = p_1^2$, $e_2 = p_2^2$, and $e_3 = p_3^2$ are linearly independent. Moreover, we can assume that the vectors $e_4 = f_1(p_2^2)$ and $e_5 = f_1(p_3^2)$ together with e_1 span Π_1. And then the vectors $e_1, \ldots, e_5$ are linearly independent.
Observe that $\Pi_2 = p_2 S$ is the plane in $\mathcal{F}$ through p_2^2 distinct from Π_0 and let $f_2 \colon x^2 \longmapsto p_2 x$ be the suitable collineation of Π_0 onto Π_2. The relation $q_3^2 \circledcirc f_1(q_3^2) = f_2(q_3^2)$ holds in $\mathfrak{M}$, but, for each $\Pi_0 \ni x^2 \neq q_3^2$ the points $x^2, f_1(x^2), f_2(x^2)$ are not collinear in $\mathfrak{M}$ and therefore they should span in $\mathfrak{P}$ the 'second plane through x^2'; we denote this plane by V_x. So, $x^2 \in V_x \in \mathcal{F}$, $V_x \neq \Pi_0$.

Assume, first, that $n = 4$; then Π_2 is spanned by e_2, e_4, and a vector d which is a combination of $e_1, \ldots, e_5$. We have $f_2(p_1^2) = p_1 p_2 = e_4$ and $f_2(p_2^2) = p_2^2 = e_2$. Then $f_2(p_3^2)$ is an arbitrary vector A in $\Pi_2 \setminus \overline{p_2^2, p_1 p_2}$. With the obvious relations $f_i(a \circledcirc b) = f_i(a) \circledcirc f_i(b)$ for $i = 1, 2$ and $a, b \in \Pi_0$ we obtain explicit formulas defining f_1 and f_2 and, finally, we can enumerate the elements of every plane V_x with $x \in \{r, p_3, q_2, q_1\}$. It was verified with the help of MapleV5 that $|V_x \cap \Pi_i| = 1$ for $x \in \{r, p_3, q_2, q_1\}$ and $i = 0, 1, 2$, but $|V_{x'} \cap V_{x''}| \neq 1$ for some distinct $x', x'' \in \{r, p_3, q_2, q_1\}$ holds for every choice of d, A. So, one cannot find within $PG(4,2)$ any suitable family $\mathcal{F}$ of planes.

Next, let $n \geq 5$. Then, one can assume that $\Pi_2 = \langle e_2, e_4, e_6 \rangle$ so as $\{e_1, \ldots, e_6\}$ is a linearly independent set in $\mathbb{V}$. Again, a Maple program computed that the obtained family of planes $\Pi_0, \Pi_1, \Pi_2, V_{q_1}, V_{q_2}, V_{p_3}, V_r$ pairwise intersect in single points, and 21 points are so obtained. We need to find the plane V_{q_3}. It is spanned by q_3^2, $p_2 q_3$ and a point B outside the planes constructed so far. Since V_{q_3} crosses these planes, without loss of generality we can assume that $B \in \langle e_1, \ldots, e_6 \rangle$. Finally, Maple program has verified that there are 4 points B such that the plane $\langle q_2^2, p_2 q_3, B \rangle$ of $PG(n,2)$ crosses each of the previously constructed planes in a single point each (besides, the resulting plane is unique!). However, we obtain only 5 new intersection points for each admissible choice of B. Since $\mathfrak{M}$ has 28 points, we could not find in $\mathfrak{P}$ a family $\mathcal{F}$ of 8 planes suitably intersecting each the other. This completes the proof of Thm. 16.

Now, note that the points in $S' = \{p_1, p_2, p_3, q_1, q_2, r\}$ yield the Veblen configuration $\mathfrak{V}$. Let us construct a realization of $\mathbf{V}_2(\mathfrak{V})$ so as each of the seven

Veblen subconfigurations ${S'}^2$ and aS' with $a \in S'$ lies on a plane of $PG(5,2)$. So, we need to find a family $\mathcal{F}'$ of planes in $PG(5,2)$ that pairwise intersect in (distinct) points. Consider the family $\mathcal{F}' = \{\Pi_0, \Pi_1, \Pi_2, V_{q_1}, V_{q_2}, V_{p_3}, V_r\}$ computed above for $n \geq 5$. Write $V_{p_1} = \Pi_1$, $V_{p_2} = \Pi_2$ and define the injection ϕ on $\mathfrak{y}_2(S')$ so as $\phi(a^2) \in \Pi_0 \cap \Pi_a$, $\phi(ab) \in V_a \cap V_b$ for $a, b \in S'$, $a \neq b$. With MapleV we could compute explicitly the coordinates of the obtained points of $PG(5,2)$. After that a computer program verified that ϕ preserves the lines of $\mathbf{V}_2(\mathfrak{V})$ and thus it is an embedding required in Prop. 1. □

The result of Thm. 16 does not mean that $\mathbf{V}_2(PG(2,2))$ cannot be realized on a plane $PG(2,q)$ for some $q > 4$. If a realization of this type exists then, necessarily, $q = 2^m$ for some $m > 2$. Any triangle p_1, p_2, p_2 of $PG(2, 2^m)$ with points $q_2 \in \overline{p_1, p_3}$, $q_3 \in \overline{p_1, p_2}$ can be uniquely completed to a Fano subplane. However, we could neither prove nor disprove the existence of a family $\mathcal{F}$ as in the proof of Thm. 16 obtained from suitable 8 triangles.

In view of Thm. 14 only the following structures:

> structures of the form $\mathbf{V}_2(\mathfrak{M})$, where $\mathfrak{M}$ is projectively realized and has at least 3-element lines and
> structures $\mathbf{V}_3(\mathfrak{M})$ for a projectively realizable partial Steiner triple system $\mathfrak{M}$

may be suspected to be embeddable into a Desarguesian projective space. Then Thm. 15 shows that there are limitations: $\mathbf{V}_3(PG(n,2))$ is not realizable for any $n > 1$. The question whether Veronese space $\mathbf{V}_3(\mathfrak{M})$ associated with any the following commonly known structures $\mathfrak{M}$

a) $AG(n,3)$,
b) the Veblen configuration,
c) the 'Mitre' configuration, or
d) the Pappus configuration

has a projective realization still remains undecided.

References

1. I. Golonko, M. Prażmowska, K. Prażmowski, *Adjacency in generalized projective Veronese spaces*, Abh. Math. Sem. Univ. Hamburg, **76** (2006), 99–114.
2. S. Kantor, *Die Konfigurationen* $(3,3)_{10}$, Sitzungsberichte Wiener Akad. **84** (1881), 1291–1314.
3. N. Melone, *Veronese spaces*, J. Geom. **12** (1983), 167–180.
4. A. Naumowicz, K. Prażmowski, *The geometry of generalized Veronese spaces*, Results Math., **45** (2004), 115–136.
5. M. Prażmowska, K. Prażmowski, *Combinatorial Veronese structures, their geometry, and problems of embeddability*, Results Math., **51** (2008), 275–308.
6. G. Tallini, *Partial linear spaces and algebraic varieties*, Symp. Math., **28** (1986), 203–217.

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The Projective Line over Finite Associative Ring with Unity

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Abstract. We discuss the projective line $\mathbb{P}(R)$ over finite associative ring R with unity.

Throughout this paper we assume that R is a finite associative ring with 1 (unity element of R). In such rings an element is either a unit or a zero divisor. The group of invertible elements of the ring R will be denoted by R^* and then the set of zero divisors is $R \setminus R^*$. Let ${}^2R$ be the two-dimensional left module over ring R. Its automorphism group is $GL_2(R)$, the general linear group of invertible two-by-two matrices with entries in R.

Definition 1 The projective line over R *is the set of free cyclic submodules of* ${}^2R$*:*

$$\mathbb{P}(R) = \{R(a,b) \subset {}^2R \colon R(a,b) \colon \text{ is a free cyclic submodule}\}.$$

Definition 2 *A pair* $(a,b) \in {}^2R$ *is* admissible*, if there exist elements* $c, d \in R$ *such that*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(R).$$

If R is commutative then the condition mentioned above is equivalent to

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^*.$$

Definition 3 *A pair* $(a,b) \in {}^2R$ *is* unimodular*, if there exist elements* $x, y \in R$ *such that*

$$ax + by = 1.$$

Unimodularity may also be defined in some other ways:

Proposition 1 *Let R be a ring and let $a, b \in R$. The following statements are equivalent:*

1. *$aR + bR = R$.*
2. *There exist elements $x, y \in R$ such that $ax + by = 1$.*
3. *There is no proper right ideal I such that $a, b \in I$.*

Proof. 1. $\Leftrightarrow$ 2.: see [1].

2. $\Rightarrow$ 3. Suppose that there exist $x, y \in R$ such that $ax + by = 1$ and let I be a right ideal such that $a, b \in I$. Obviously, $ax \in I$ for all $x \in R$ and $by \in I$ for all $y \in R$. Consequently $(ax + by) \in I$ for all $x, y \in R$. Thus $1 \in I$, and therefore $I = R$.

3. $\Rightarrow$ 2. Let $ax+by \neq 1$ for all $x, y \in R$, then $\{ax+by; x, y \in R\} = aR+bR \neq R$. aR, bR are ideals, thus $aR + bR$ is an ideal too, and it contains a, b. So, $aR + bR$ is a proper right ideal which contradicts 3.
This completes the proof of the proposition.

Remark 1 *[2] It is well known that each admissible pair $(a, b) \in {}^2R$ is unimodular.*

What about the converse implication?

There are examples of rings where unimodularity does not imply admissibility [3, Remark 5.1].

Rings with the property $ab = 1 \Rightarrow ba = 1$ are called *Dedekind-finite*. Now we have the following

Proposition 2 [2] *Let R be a ring (not necessary finite). Then the following three statements are equivalent:*

1. *R is Dedekind-finite.*
2. *If a pair (a, b) is unimodular then it is admissible.*
3. *No point of $\mathbb{P}(R)$ is properly contained in another point of $\mathbb{P}(R)$.*

It follows then that if R is a finite or commutative ring, then admissibility and unimodularity are equivalent. Rings of stable rank 2 (for example: local rings and matrix rings over fields) satisfy this property as well.

Remark 2 *Obviously, if $(a, b) \in {}^2R$ is unimodular then $R(a, b)$ is a free cyclic submodule of ${}^2R$.*

In certain cases free cyclic submodules may also be generated by *non-admissible pairs*, but if R is a local ring, then every free cyclic submodule is represented by unimodular pair [6, Theorem 20 1.].

Corollary 1 *For a local ring R, the projective line may over R be equivalently defined as:*

1. $\mathbb{P}(R) = \{R(a, b) \subset {}^2R \colon (a, b)$ *is an unimodular pair*$\}$.
2. $\mathbb{P}(R) = \{R(u, x) \colon x \in R\} \cup \{R(d, u) \colon d \in I\}$, *$u$ - a fixed element of R^*, I - the maximal ideal of R.*

Example The projective line over ring of residue classes modulo four.

$$\mathbb{P}(Z_4) = \{Z_4(1, 0),\ Z_4(1, 1),\ Z_4(1, 2),\ Z_4(1, 3),\ Z_4(0, 1),\ Z_4(2, 1)\}$$

Note that every point of the projective line over ring Z_4 is generated by a unimodular pair and each of that pairs has at least one entry which is a unit. This property stays true for the projective line over an arbitrary local ring.

Projective line $\mathbb{P}(R)$ carries two mutually complementary relations: distant "Δ" and parallel "$\not\Delta$". They are not trivial if and only if R is not a field. Hence the projective line can be defined as the set of points with relation distant: $\mathbb{P}(R, \Delta)$ or parallel: $\mathbb{P}(R, \not\Delta)$.

Definition 4 *Two points $R(a,b)$, $R(c,d)$ of the projective line $\mathbb{P}(R)$ are called* distant, *if*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(R).$$

Otherwise, they are called parallel *(or* neighbour*).*

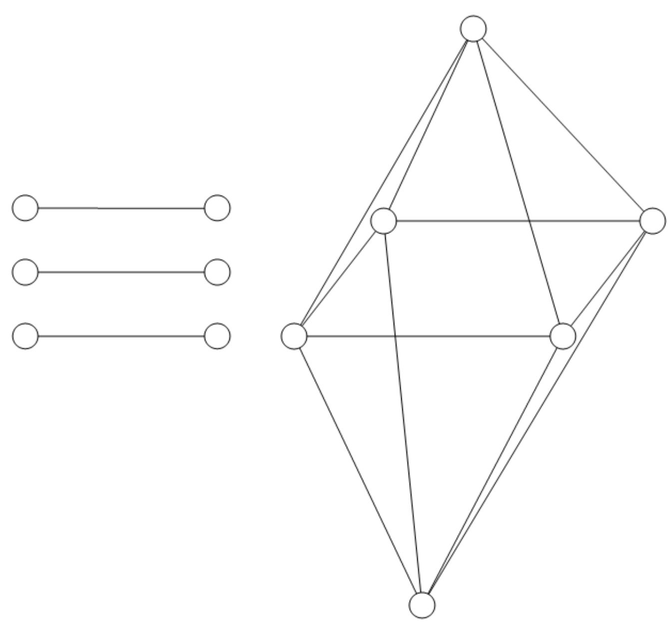

Fig. 1. Parallel (left) and distant (right) graph on the projective line over Z_4.

By an automorphism of a projective line we mean an automorphism of the distant/parallel graph on the corresponding projective line $\mathbb{P}(R, \Delta)/\mathbb{P}(R, \not\Delta)$.

There is also another relation of a parallelism defined on a projective line: radical parallelism (denoted by $\|$).

Definition 5 *[4, 3.1.] Two points $R(a,b)$, $R(c,d)$ of the projective line $\mathbb{P}(R)$ are called* radical parallel, *if the set of the points distant to $R(a,b)$ and the set of the points distant to $R(c,d)$ coincide:*

$$R(a,b) \,||\, R(c,d) \Leftrightarrow \Delta(R(a,b)) = \Delta(R(c,d)).$$

($\Delta(R(a,b))$ is the neighbourhood of $R(a,b)$ in the distant graph.)

For local rings parallelism and radical parallelism are equivalent [4, 3.5]. We use the term 'parallelism' in the first meaning.

The group $GL_2(R)$ acts on the projective line and

$$\bigwedge_{\varphi \in GL_2(R)} \bigwedge_{p,q \in \mathbb{P}(R)} p\Delta q \Leftrightarrow \varphi(p)\Delta\varphi(q).$$

The elementary linear group $E_2(R)$ is the normal subgroup of $GL_2(R)$ generated by all matrices of the form

$$E(x) = \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix}, x \in R.$$

The elementary linear group $E_2(R)$ and the set of all two-by-two diagonal matrices over R generate the group $GE_2(R)$, which is also a subgroup of the general linear group $GL_2(R)$.

Definition 6 *Ring R is called a GE_2-*ring *if $GL_2(R) = GE_2(R)$.*

In Theorem 1 and Corollary 2 we will be concerned with the classical projective line (its points are represented by admissible pairs only).

Theorem 1. *[3, Theorem 3.2 (d)] The distant graph on the projective line $G\big(\mathbb{P}(R,\Delta)\big)$ is connected if, and only if, R is a GE_2-ring.*

Theorem 2. *[1] Let J be the Jacobson radical of R. If R/J satisfies condition minimum for left ideals then the ring R is a GE_2-ring.*

From Theorems 1 and 2 we obtain:

Corollary 2 *The distant graph on the projective line over a finite ring R is connected.*

Next we will characterize the projective line over a finite commutative ring and its automorphism group. We shall need the following:

Theorem 3. *[4, 6.1.] Let R be the direct product of rings R_i, $i \leqslant k$, i.e. $R = R_1 \times R_2 \times ...R_k$. Then:*

$$\mathbb{P}(R,\Delta) \simeq \mathbb{P}(R_1,\Delta) \times \mathbb{P}(R_2,\Delta) \times ... \times \mathbb{P}(R_k,\Delta.$$

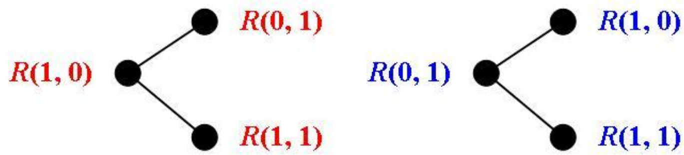

Fig. 2. Example of distant points of $\mathbb{P}(Z_2)$.

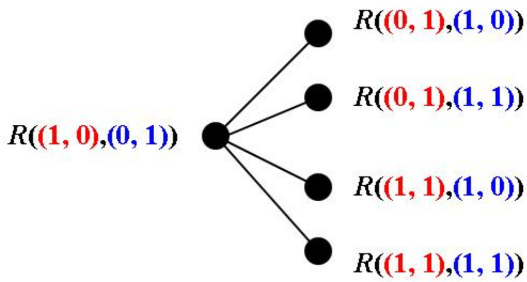

Fig. 3. Example of distant points of $\mathbb{P}(Z_2 \times Z_2)$.

Another way to state this is to say:

$$G\big(\mathbb{P}(R,\Delta)\big) \simeq G\big(\mathbb{P}(R_1,\Delta)\big) \times G\big(\mathbb{P}(R_2,\Delta)\big) \times ... \times G\big(\mathbb{P}(R_k,\Delta)\big),$$

where $'\times'$ denotes standard cartesian product.

Theorem 4. *[5, VI.2] Every finite commutative ring R is the direct product of local rings.*

It follows then that the description of the projective line over finite commutative ring can be based on the projective line over local rings (compare, e.g. Figure 4).

Now we will be concerned with the automorphism group of the parallel/distant graph on the projective line over an arbitrary local ring. We shall need the following remark.

Remark 3 *Let R be a field and let $f : R \longrightarrow R$ be a bijection. If there is a well defined transformation $\phi(f) : \mathbb{P}(R) \longrightarrow \mathbb{P}(R)$ characterized by the condition $\phi(f)(R(a,b)) = R(f(a), f(b))$ then $f(0) = 0$ and $f|_{R^*} : R^* \to R^*$ and $f = \alpha c$, where α is an automorphism of R^*, $c \in R^*$.*
In that case we say that $\phi(f)$ is an automorphism induced by the bijection f.

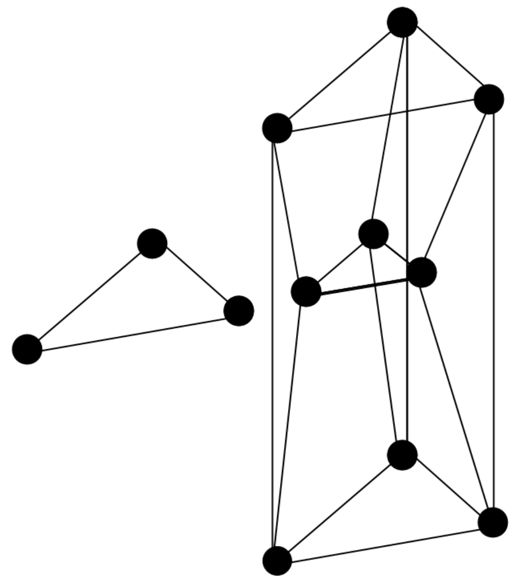

Fig. 4. The distant graphs: on $\mathbb{P}(Z_2)$ (left) and on $\mathbb{P}(Z_2 \times Z_2)$ (right).

If R is a finite field $(|R| = n,\ |\mathbb{P}(R)| = n+1)$ then $|Aut(\mathbb{P}(R))| = (n+1)!$.
Any automorphism of $\mathbb{P}(R)$ over a local ring R comes from some automorphism of $\mathbb{P}(R/I)$, hence we can characterize any such automorphism of $\mathbb{P}(R)$, for a local ring R.

Let us denote by Z_n the ring of residue classes modulo n. In the case of rings Z_{p^n} (p - prime) we can explicitly compute the cardinality of $\mathbb{P}(Z_n)$, of $\mathrm{Aut}\big(\mathbb{P}(Z_{p^n})$, and of $PGL_2(Z_{p^n})$.

By Corollary 1:

$$\mathbb{P}(Z_{p_i^{n_i}}) = \{Z_{p_i^{n_i}}(u,x); x \in Z_{p_i^{n_i}}\} \cup \{Z_{p_i^{n_i}}(d,u); d \in I\},$$

where u is a fixed element of $Z^*_{p_i^{n_i}}$ and I is the maximal ideal of $Z_{p_i^{n_i}}$.

Then it follows that

$$|\mathbb{P}(Z_{p_i^{n_i}})| = |Z_{p_i^{n_i}}| + |I| = p_i^{n_i} + p_i^{n_i - 1}.$$

Corollary 3 *If $Z_n = Z_{p_1^{n_1}} \times Z_{p_2^{n_2}} \times ... \times Z_{p_k^{n_k}}$ where p_i are distinct primes for $i = 1, 2, ..., k$), then:*

$$\begin{aligned} \mid \mathbb{P}(Z_n) \mid =& \mid \mathbb{P}(Z_{p_1^{n_1}}) \times \mathbb{P}(Z_{p_2^{n_2}}) \times ... \times \mathbb{P}(Z_{p_k^{n_k}}) \mid = \\ &= \mid \mathbb{P}(Z_{p_1^{n_1}}) \mid \cdot \mid \mathbb{P}(Z_{p_2^{n_2}}) \mid \cdot ... \cdot \mid \mathbb{P}(Z_{p_k^{n_k}}) \mid = \\ &= (p_1^{n_1} + p_1^{n_1-1}) \cdot (p_2^{n_2} + p_2^{n_2-1}) \cdot ... \cdot (p_k^{n_k} + p_k^{n_k-1}). \end{aligned}$$

and

$$|Aut(\mathbb{P}(Z_{p^n}))| = (p+1)! \cdot (p^{n-1}!)^{p+1},$$

(in particular $|Aut(\mathbb{P}(Z_p))| = (p+1)!$*) where*

$p+1$ *is the number of connected components of the parallel graph on* $\mathbb{P}(Z_{p^n})$ *(on* $\mathbb{P}(Z_p)$*, resp.),*

p^{n-1} *is the number of points in any connected component, also the number of zero divisors of the ring* Z_{p^n}*.*

The general linear group $GL_2(R)$ induces the subgroup of general automorphism group $Aut(\mathbb{P}(R))$ on the projective line $\mathbb{P}(R)$, namely the projective linear group of the ring R denoted by $PGL_2(R)$. This is the group of invertible two-by-two matrices with entries in R, accurate to proportionality: $PGL_2(R) = GL_2(R)/\sim$.

Hence

$$\mid PGL_2(R) \mid = \mid GL_2(R)/\sim \mid = \mid GL_2(R) \mid / \mid R^* \mid .$$

Remark 4 *[7]* $|GL_2(Z_p)| = (p^2-1)(p^2-p)$

Corollary 4 $|PGL_2(Z_p)| = |GL_2(Z_p)/\sim| = (p^2-1)(p^2-p)/(p-1) =$

$$= (p+1)(p^2-p) \leq (p+1)! \ = |Aut(\mathbb{P}(Z_p))|$$

Remark 5 *[7]* $|GL_2(Z_{p^n})| = p^4|GL_2(Z_{p^{n-1}})| = p^{4\cdot 2}|GL_2(Z_{p^{n-2}})| = ... =$

$$= p^{4(n-1)}|GL_2(Z_{p^{n-(n-1)}})| = p^{4(n-1)}|GL_2(Z_p)|$$

Corollary 5 $|PGL_2(Z_{p^n})| = |GL_2(Z_{p^n})/\sim| = p^{4(n-1)}|GL_2(Z_p)/\sim| =$

$$= p^{4(n-1)}(p+1)(p^2-p) \ \leq (p+1)! \cdot (p^{n-1}!)^{p+1} = |Aut(\mathbb{P}(Z_{p^n}))|$$

The description of the projective lines over rings of order p^i, $i \leq 3$ is known. For $i=3$ we have exactly one noncommutative ring ($T_2(GF(2))$ - the ring of triangular matrices over $GF(2)$ [8]), and the other are local rings. The projective line over every finite commutative ring and rings of order p^i ($i \leq 3$, p - prime) can be described by Corollary 1 and Theorem 4. The corresponding automorphism group of the projective line can be determined by Remark 3.

As we mentioned before, e.g. in the projective line $\mathbb{P}(T_2(GF(2)))$, there are free cyclic submodules generated by non-admissible pairs. These pairs have been called outliers. They are not contained in any free cyclic submodule generated by an admissible pair. There are some examples of finite rings with outliers (see Fig. 5) and examples of automorphism group of $\mathbb{P}(R)$ in this case are known.

Example Non-unimodular part of the projective line $\mathbb{P}(T_2(GF(2)))$ consists of three outliers, namely of the pairs of matrices with nonzero first kolumn:

$$(\begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}), \quad (\begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}), \quad (\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix})$$

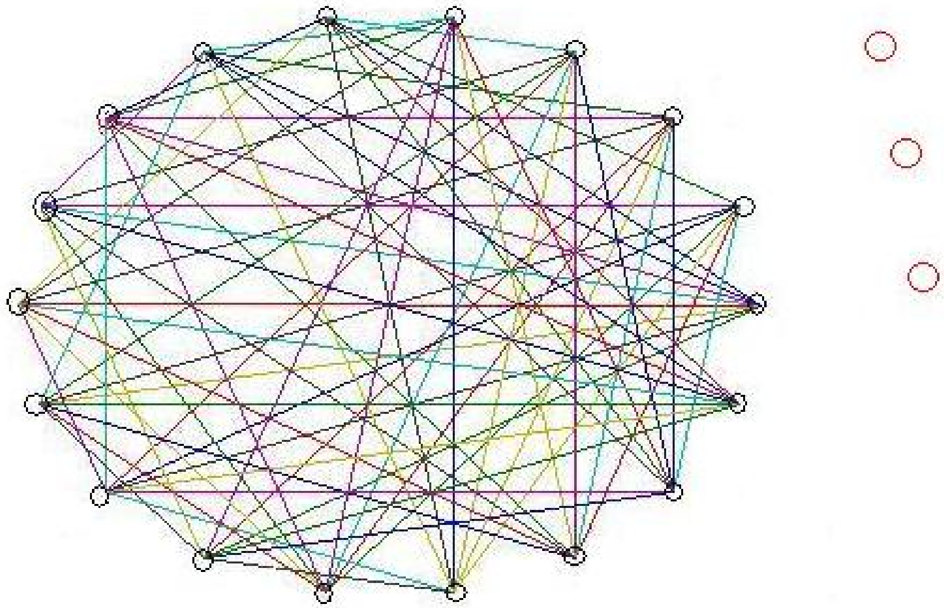

Fig. 5. The distant graph on $\mathbb{P}(T_2(GF(2)))$.

Outliers are not distant to any other point, therefore they are isolated points in the distant graph of an arbitrary projective line.

In the general case of the projective line over an *arbitrary* finite ring the description of the automorphism group is more complex. To this aim the following can be used.

Theorem 5. *[5, I.1] Every finite ring is isomorphic to direct sum of rings of prime power order.*

Corollary 6 *The projective line over a finite ring can be described via characterizations of the projective lines over rings of prime power order.*

References

1. Bass H., *K-theory and stable algebra*, Publications mathématiques de l'I.H.E.S, tom 22, 1964.
2. Blunck A., Havlicek H., *Projective Representations I. Projective lines over rings*, Abh. Math. Sem. Univ. Hamburg, 2000.
3. Blunck A., Havlicek H, *The Connected Components of the Projective Line over a Ring*, Adv. Geom. 1 (2001), 107-117.

4. Blunck A., Havlicek H., *On distant-isomorphisms of projective lines*, Aequationes Math. 69 (2005) 146-163.
5. McDonald B.R., *Finite Rings with identity*, New York: Marcel Dekker; 1974.
6. Hall J.L., Saniga M., *Free Cyclic Submodules and Non-Unimodular Vectors*, arXiv:1107.3050v2 [math.CO].
7. Han J., *The General Linear Group over a Ring*, Bull. Korean Math. Soc. 43(2006), (619-626).
8. Havlicek H., Matraś A., Pankov M., *Geometry of free cyclic submodules over ternions*, Abh. Math. Sem. Univ. Hamburg (2011).

Part III

Differential Equations

Recent Results in Pure and Applied Mathematics, Podlasie 2014
by A. Gomolińska, A. Grabowski, M. Hryniewicka, M. Kacprzak, E. Schmeidel (Eds.)

Comparison of Boundedness of Solutions of Differential and Difference Equations

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Abstract. In mathematics and computer science, an algorithm is a procedure (a finite set of well-defined instructions) for accomplishing some task which, given an initial state, will terminate in a defined end-state. Such algorithm usually is a recurrence relation. The difference equation, is an equation which defines a sequence recursively: each term of the sequence is defined as a function of the preceding terms. Most algorithms can be directly implemented by computer programs; any other algorithms can at least in theory be simulated by computer programs. In the this paper we are looking on the differences between asymptotic properties of solutions of the difference equation and its continuous analogies on example of third order linear homogeneous differential and difference equations with constant coefficients. We take the both equations with the same characteristic equation. We show that these equations (differential and difference) can have solutions with different properties concerning boundedness.
Keywords: differential equation, difference equation, recurrence, linear, third order, bounded solution
AMS Subject classification: 34A05, 34C11, 39A10

1 Introduction

Together with the development of modern computational methods, discrete equations, so-called difference equations, have acquired a great importance. They replace often differential equations in mathematical modelling because they are the ready algorithms for computer programs. For example, logistic equation describing increase of population is written in the form of differential equation $y' = y(b - ay)$ (Verhulst-Pearl equation) or in the form of difference equation $u_{n+1} = u_n(b - au_n)$ (Pielou equation).

In mathematical modelling, there are many examples proving that continuous and discrete models are created in such way that the derivatives are replaced by suitable terms of the sequence. For example, recently Yue-Wen Cheng and Hui-Sheng Ding studied in [3] a linear Volterra difference equation

$$x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^{n} \int_0^t K_{ij}(t,s)x_j(s)\mathrm{ds}$$

which is considered as a continuous analogue of Volterra difference equation

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i)$$

investigated by Diblík and Schmeidel in [4].

It appears that application of a discrete mathematical model instead of the continuous one can lead to qualitative different solutions, particularly in respect of oscillation or boundedness. Unlike quality of solutions can be seen even in the case of linear equations with constant coefficients. In [14] we show that these equations (differential and difference) can have solutions with different properties concerning oscillation. In this paper we analyse boundedness.

We consider third order linear homogeneous equation with constant coefficients written in the continuous and discrete form, namely the differential equation

$$\frac{d^3 f}{dt^3} + a\frac{d^2 f}{dt^2} + b\frac{df}{dt} + cf = 0 \tag{1}$$

and the difference equation

$$u_{n+3} + au_{n+2} + bu_{n+1} + cu_n = 0, \tag{2}$$

where $f : R \to R$, $u : N \to R$, R denotes the set of real numbers, N the set of positive integers and a, b, c are real constants.

A function $f : R \to R$ (a sequence $u : N \to R$) is called trivial if there exists a $t_0 \in R$ (an $n_0 \in N$) such that $f(t) = 0$ for every $t > t_0$ ($u_n = 0$ for every $n > n_0$). Otherwise a function f (a sequence u) is said to be nontrivial. Every nontrivial real function f (sequence u) satisfying equation (1) ((2)) is called a solution of this equation.

A solution of equation (1) ((2)) is called bounded if it is a bounded function (sequence) on the real set $[0, \infty)$ (N). Otherwise a solution is said to be unbounded. Equation (1) or (2) is called bounded (unbounded) if every its solution is bounded (unbounded). We will consider linear independent solutions composing a fundamental set.

Fundamentals of differential equations theory can be found in many monographs. The background of difference equation theory is given by Agarwal [1], Elaydi [6], Kelley and Peterson [8].

Particularly, great attention has been paid to the study of third order differential equations, because these equations describe many mathematical models of huge interest in engineering, biology and physics. Properties of third order nonlinear difference equation were investigated among many others by Seshader and Smita. There is a monograph written by this authors [15] in which an entire chapter is devoted to the third order linear differential equations with constant coefficients. Special attention is paid to the oscillatory properties of solutions.

Boundedness of solutions of third order difference equations was considered for example by Andruch-Sobiło and Migda [2], Došla and Kobza [5], Graef and Thandapani [7], Smith [16], Smith and Taylor [18] and oscillation by Kobza [9], Migda, Schmeidel and Drozdowicz [10], Popenda and Schmeidel [13], Saker [14], Smith [17], Thandapani and Mahalingam [19].

Putting $f(t) = \exp(rt)$ in equation (1) or $u_n = r^n$ in equation (2) the same characteristic equation

$$r^3 + ar^2 + br + c = 0 \tag{3}$$

is obtained in the continuous as well as in the discrete case. Solutions of equation (3) are described in [12].
Let us denote

$$q = c - \frac{1}{3}ab + \frac{2}{27}a^3, \quad \Delta = c^2 + \frac{4}{27}b^3 - \frac{2}{3}abc - \frac{1}{27}a^2b^2 + \frac{4}{27}a^3c. \tag{4}$$

We will use the following theorem, which can be found in [14].

Theorem 1. *(i) If $\Delta > 0$, then (3) has one real root r_1 and two complex conjugate roots r_2, r_3 with non-vanishing imaginary parts:*

$$\begin{aligned}
r_1 &= \sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}} - \tfrac{a}{3}, \\
r_2 &= -\tfrac{1}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right) - \tfrac{a}{3} \\
&\quad + i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right), \\
r_3 &= -\tfrac{1}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right) - \tfrac{a}{3} \\
&\quad - i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right).
\end{aligned} \tag{5}$$

(ii) If $\Delta = 0$, then (3) has three real roots given by (5), but one is multiple:

$$r_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a}{3}, \quad r_2 = r_3 = \sqrt[3]{\frac{q}{2}} - \frac{a}{3}.$$

(iii) If $\Delta < 0$, then (3) has three different real roots:

$$r_{k+1} = \sqrt[6]{16(q^2 - \Delta)} \cos \frac{\arccos \frac{-q}{\sqrt{q^2-\Delta}} + 2k\pi}{3} - \frac{a}{3}, \quad k = 0, 1, 2.$$

Remark 1. The following equality $q^2 - \Delta = \frac{4}{27}(\frac{1}{27}a^6 - \frac{1}{3}a^4b + a^2b^2 - b^3)$ holds.

Set

$$\begin{aligned} d &= \sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}} \quad \text{for} \quad \Delta \geqslant 0, \\ p_k &= \cos \frac{\arccos \frac{-q}{\sqrt{q^2-\Delta}} + 2k\pi}{3} \quad \text{for} \quad \Delta < 0 \quad (k = 0, 1, 2). \end{aligned} \tag{6}$$

2 Differential equations

Let us consider differential equation (1) as well as q, Δ defined by (4) and d, p_k by (2).

Theorem 2. *Let $a \geqslant 0$ and $\Delta > 0$.*
(i) If

$$-\frac{2}{3}a \leqslant d \leqslant \frac{1}{3}a, \tag{7}$$

then (1) is bounded.
(ii) If

$$d < -\frac{2}{3}a \quad \text{or} \quad d > \frac{1}{3}a, \tag{8}$$

then (1) has bounded and unbounded solutions.

Proof. Since (7), by (5) we have $r_1 = d - \frac{1}{3}a \leqslant 0$ and $\mathrm{Re}\ r_2 = \mathrm{Re}\ r_3 = -\frac{1}{2}d - \frac{1}{3}a \leqslant 0$. Hence (1) has only bounded solutions. From the first inequality of (8), by (5) we have $r_1 < -a \leqslant 0, \quad \mathrm{Re}\ r_2 = \mathrm{Re}\ r_3 > 0$. From the second inequality of (8) we get $r_1 > 0, \quad \mathrm{Re}\ r_2 = \mathrm{Re}\ r_3 < -\frac{1}{2}a \leqslant 0$. In the both above cases equation (1) has bounded and unbounded solutions.

Theorem 3. *Let $a \geqslant 0$ and $\Delta = 0$.*
(i) If

$$-\frac{2}{3}a < d \leqslant \frac{1}{3}a,$$

then (1) is bounded.
(ii) If

$$d \leqslant -\frac{2}{3}a \quad \text{or} \quad d > \frac{1}{3}a,$$

then (1) has bounded and unbounded solutions.

Proof. The proof of this theorem is analogous to that of Theorem 2 and hence omitted.

Theorem 4. *Let $a < 0$ and $\Delta \geqslant 0$.*
(i) If

$$\frac{1}{3}a < d < -\frac{2}{3}a, \tag{9}$$

then (1) is unbounded.
(ii) If

$$d \leqslant \frac{1}{3}a \quad \text{or} \quad d \geqslant -\frac{2}{3}a, \tag{10}$$

then (1) has bounded and unbounded solutions.

Proof. Since (9), by (5) we have $r_1 > 0, \quad \mathrm{Re}\ r_2 = \mathrm{Re}\ r_3 > 0$. Hence (1) has only unbounded solutions. From the first inequality of (10) we get $r_1 \leqslant 0, \quad \mathrm{Re}\ r_2 = \mathrm{Re}\ r_3 \geqslant -\frac{1}{2}a > 0$. From the second inequality of (10) we obtain $r_1 \geqslant -a > 0, \quad \mathrm{Re}\ r_2 = \mathrm{Re}\ r_3 \leqslant 0$. Thus in these cases equation (1) has bounded and unbounded solutions.

Theorem 5. *Let $\Delta < 0$.*

(*i*) *If*

$$p_k \leqslant \frac{a}{3\sqrt[6]{16(q^2-\Delta)}}, \quad k = 0, 1, 2, \tag{11}$$

then (1) *is bounded.*

(*ii*) *If*

$$p_k > \frac{a}{3\sqrt[6]{16(q^2-\Delta)}}, \quad k = 0, 1, 2, \tag{12}$$

then (1) *is unbounded.*

(*iii*) *In the other cases equation* (1) *has bounded and unbounded solutions.*

Proof. From (11) we have $r_k \leqslant 0$ for $k = 1, 2, 3$, and from (12) we obtain $r_k > 0$ for $k = 1, 2, 3$. The proof is completed.

3 Difference equations

Now let us consider difference equation (2). Assume $c \neq 0$, what is required since (2) is a third order equation. Assume also q, Δ, d, p_k are defined as in part 2 of this paper. Set

$$D = \sqrt{d^2 + \frac{1}{3}ad + \frac{1}{9}a^2 - \frac{3}{2}\sqrt[3]{2(q^2-\Delta)}}.$$

Theorem 6. *Let $\Delta > 0$.*

(*i*) *If*

$$max\ (|d - \frac{1}{3}a|, D) \leqslant 1, \tag{13}$$

then (2) *is bounded.*

(*ii*) *If*

$$min\ (|d - \frac{1}{3}a|, D) > 1, \tag{14}$$

then (2) *is unbounded.*

(*iii*) *In the other cases equation* (2) *has bounded and unbounded solutions.*

Proof. Since (13), by (5) we have $|r_1| \leqslant 1$. Moreover

$$|r_2| = |r_3| = \sqrt{\frac{1}{4}d^2 + \frac{1}{3}ad + \frac{1}{9}a^2 + \frac{3}{4}\left(d^2 - 4\sqrt[3]{\frac{q^2-\Delta}{4}}\right)}$$

$$= \sqrt{d^2 + \frac{1}{3}ad + \frac{1}{9}a^2 - 3\sqrt[3]{\frac{q^2 - \Delta}{4}}} = D \leqslant 1.$$

Hence the equation (2) has only bounded solutions.Since (14), then $|r_1| > 1$ and $|r_2| = |r_3| > 1$. Thus equation (2) has only unbounded solutions. When (13) and (14) don't hold, equation (2) has bounded and unbounded solutions. This completes the proof.

Theorem 7. *Let $\Delta = 0$.*

(*i*) *If*

$$|d - \frac{1}{3}a| \leqslant 1, \quad D < 1,$$

then (2) *is bounded.*

(*ii*) *If*

$$min\ (|d - \frac{1}{3}a|, D) > 1,$$

then (2) *is unbounded.*

(*iii*) *In the other cases equation* (2) *has bounded and unbounded solutions.*

Proof. The proof of this theorem is analogous to that of Theorem 6 and hence omitted.

Theorem 8. *Let $\Delta < 0$.*

(*i*) *If*

$$\frac{-3 + a}{3\sqrt[6]{16(q^2 - \Delta)}} \leqslant p_k \leqslant \frac{3 + a}{3\sqrt[6]{16(q^2 - \Delta)}} \quad \textit{for} \quad k = 0, 1, 2 \tag{15}$$

then (2) *is bounded.*

(*ii*) *If*

$$\begin{array}{ll} p_k < \frac{-3+a}{3\sqrt[6]{16(q^2-\Delta)}} & \textit{for} \quad k = 0, 1, 2 \\ \textit{or} & \\ p_k > \frac{3+a}{3\sqrt[6]{16(q^2-\Delta)}} & \textit{for} \quad k = 0, 1, 2, \end{array} \tag{16}$$

then (2) *is unbounded.*

(*iii*) *In the other cases equation* (2) *has bounded and unbounded solutions.*

Proof. In the case of (15) we have $|r_k| \leqslant 1$ for $k = 1, 2, 3$, and in the cases of (16) we get $|r_k| > 1$ for $k = 1, 2, 3$. The proof is completed.

Remark 2. For $a \geqslant 0$ and $\Delta \geqslant 0$ equation (1) is never unbounded but equation (2) can be. It is e.g. for $a = 0$, $b = -12$, $c = 16$. Then $q = 16$, $\Delta = 0$, $d = -4$, $D = 2$ and min $(|d - \frac{1}{3}a|, D) = 2$.

Remark 3. For $a < 0$ and $\Delta \geqslant 0$ equation (1) is never bounded but equation (2) can be. It is e.g. for $a = c = -1$, $b = 1$. We obtain $q = -\frac{20}{27}$, $\Delta = \frac{16}{27}$, $d = \frac{2}{3}$, $D = 1$ and min $(|d - \frac{1}{3}a|, D) = 1$.

4 Corollaries

One can easy see that the equations (1) and (2) can have different properties. It is shown by the following examples.

Example 1. If $a = b = c = 1$, then $q = \frac{20}{27}$, $\Delta = \frac{16}{27}$, $d = -\frac{2}{3}$, $D = 1$, $\min(\left|d - \frac{1}{3}a\right|, D) = 1$. By Theorem 2(i) and Theorem 6(i), the both equations are bounded.

Example 2. If $a = -7$, $b = 16$, $c = -12$, then $q = -\frac{2}{27}$, $\Delta = 0$, $d = \frac{2}{3}$, $D = 2$, $\min(\left|d - \frac{1}{3}a\right|, D) = 2$ By Theorem 4(i) and Theorem 7(ii), the both equations are unbounded.

Example 3. If $a = 0$, $b = -7$, $c = 6$, then $q = 6$, $\Delta = -\frac{400}{27}$, $p_0 = \frac{\sqrt{21}}{7}$, $p_1 = -\frac{3\sqrt{21}}{14}$, $p_2 = \frac{\sqrt{21}}{14}$, $\frac{a}{3\sqrt[6]{16(q^2-\Delta)}} = 0$, $\frac{-3+a}{3\sqrt[6]{16(q^2-\Delta)}} = -\frac{\sqrt{21}}{14}$, $\frac{3+a}{3\sqrt[6]{16(q^2-\Delta)}} = \frac{\sqrt{21}}{14}$. By Theorem 5(iii) and Theorem 8(iii), the both equations have bounded and unbounded solutions.

Example 4. If $a = 7$, $b = 16$, $c = 12$, then $q = \frac{2}{27}$, $\Delta = 0$, $d = -\frac{2}{3}$, $D = 2$, $\min(\left|d - \frac{1}{3}a\right|, D) = 2$. By Theorem 3(i), the equation (1) is bounded, and by Theorem 7(ii), the equation (2) is unbounded.

Example 5. If $a = c = 6$, $b = 11$, then $q = 0$, $\Delta = -\frac{4}{27}$, $p_0 = \frac{\sqrt{3}}{2}$, $p_1 = -\frac{\sqrt{3}}{2}$, $p_2 = 0$, $\frac{a}{3\sqrt[6]{16(q^2-\Delta)}} = \sqrt{3}$, $\frac{-3+a}{3\sqrt[6]{16(q^2-\Delta)}} = \frac{\sqrt{3}}{2}$, $\frac{3+a}{3\sqrt[6]{16(q^2-\Delta)}} = \frac{3}{2}\sqrt{3}$. By Theorem 5(i), the equation (1) is bounded, and by Theorem 8(iii), the equation (2) has bounded and unbounded solutions.

Example 6. If $a = -\frac{11}{6}$, $b = 1$, $c = -\frac{1}{6}$, then $q = -\frac{35}{2916}$, $\Delta = -\frac{1}{8748}$, $p_0 = \frac{7}{2\sqrt{13}}$, $p_1 = -\frac{5}{2\sqrt{13}}$, $p_2 = -\frac{1}{\sqrt{13}}$, $\frac{a}{3\sqrt[6]{16(q^2-\Delta)}} = -\frac{11}{2\sqrt{13}}$, $\frac{-3+a}{3\sqrt[6]{16(q^2-\Delta)}} = -\frac{29}{2\sqrt{13}}$, $\frac{3+a}{3\sqrt[6]{16(q^2-\Delta)}} = \frac{7}{2\sqrt{13}}$. By Theorem 5(ii), the equation (1) is unbounded, and by Theorem 8(i), the equation (2) is bounded.

Example 7. If $a = c = -6$, $b = 11$, then $q = 0$, $\Delta = -\frac{4}{27}$, $p_0 = \frac{\sqrt{3}}{2}$, $p_1 = -\frac{\sqrt{3}}{2}$, $p_2 = 0$, $\frac{a}{3\sqrt[6]{16(q^2-\Delta)}} = -\sqrt{3}$, $\frac{-3+a}{3\sqrt[6]{16(q^2-\Delta)}} = -\frac{3}{2}\sqrt{3}$, $\frac{3+a}{3\sqrt[6]{16(q^2-\Delta)}} = -\frac{\sqrt{3}}{2}$. By Theorem 5(ii), the equation (1) is unbounded, and by Theorem 8(iii), the equation (2) has bounded and unbounded solutions.

Example 8. If $a = c = -1$, $b = 1$, then $q = -\frac{20}{27}$, $\Delta = \frac{16}{27}$, $d = \frac{2}{3}$, $D = 1$, $\min(\left|d - \frac{1}{3}a\right|, D) = 1$. By Theorem 4(ii), the equation (1) has bounded and unbounded solutions, and by Theorem 6(i), the equation (2) is bounded.

Example 9. If $a = 0$, $b = -12$, $c = 16$, then $q = 16$, $\Delta = 0$, $d = -4$, $D = 2$, $\min(\left|d - \frac{1}{3}a\right|, D) = 2$. By Theorem 3(ii), the equation (1) has bounded and unbounded solutions, and by Theorem 7(ii), the equation (2) is unbounded.

References

1. R.P. Agarwal, *Difference Equations and Inequalities. Theory, Methods and Applications,* Marcel Dekker, New York 2000 (second edition).
2. A. Andruch-Sobiło, M. Migda, *Bounded solutions of third order nonlinear difference equations,* Rocky Mountain Journal of Mathematics 36, 1 (2006), 23-34.
3. Y. Cheng, H. Ding, *Asymptotic behavior of solutions to a linear Volterra integrodifferential system*, Abstr. Appl. Anal. 2013, Art. ID 245905, 5 pp.
4. J. Diblík, E. Schmeidel, *On the existence of solutions of linear Volterra difference equations asymptotically equivalent to a given sequence*, Appl. Math. Comput. 218 (2012), 18, 9310-9320.
5. Z. Došla, A. Kobza, *Global asymptotic properties of third order difference equations,* Computers and Mathematics with Applications 48, 1-2 (2004), 191-200.
6. S.N. Elaydi, *An Introduction to Difference Equations,* Springer, 1996.
7. J.R. Graef, E. Thandapani, *Oscillatory and asymptotic behavior of solutions of third order delay difference equations,* Funkcialaj Ekvacioj 42, 3 (1999), 355-369.
8. W.G. Kelley, A.C. Peterson, *Difference Equations, an Introduction with Applications,* Academic Press, 1991.
9. A. Kobza, *Property A for third order difference equations*, Studies of the University of Zilina, Mathematical Series 17 (2003), 109-114.
10. M. Migda, E. Schmeidel, A. Drozdowicz, *Nonoscillation results for some third order nonlinear difference equation*, Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis Mathematica 13 (2003), 185-192.
11. J. Mikołajski, E. Schmeidel, *Comparison of properties of solutions of differential equations and recurrence equations with the same characteristic equation*, Opuscula Mathematica 26, 2 (2006), 343-349.
12. A. Mostowski, M. Stark, *Elementy algebry wyższej,* PWN, Warszawa 1968 (in Polish).
13. J. Popenda, E. Schmeidel, *Nonoscillatory solutions of third order difference equations*, Portugaliae Mathematica 49 (1992), 233-239.
14. S.H. Saker, *Oscillation of third-order difference equations*, Portugaliae Mathematica 61, 3 (2004), 249-257.
15. P. Seshadev, P. Smita, *Theory of Third Order Differential Equations*, Springer 2014.
16. B. Smith, *Oscillatory and asymptotic behavior in certain third order difference equations*, Rocky Mountain Journal of Mathematics 17, 3 (1987), 597-606.
17. B. Smith, *Oscillation and no oscillation theorems for third order quasi-adjoint difference equations*, Portugaliae Mathematica 45, 3 (1988), 229-243.
18. B. Smith, W.E. Taylor, *Oscillatory and asymptotic behavior of certain fourth order difference equations*, Rocky Mountain Journal of Mathematics 16, 2 (1986), 403-406.
19. E. Thandapani, K. Mahalingam, *Oscillatory properties of third order neutral delay difference equatons,* Demonstratio Mathematica 35 (2002), 325-337.

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Boundedness of Solutions of Neutral Type Nonlinear Difference System with Deviating Arguments

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Abstract. In this paper, we consider three–dimensional nonlinear difference system with deviating arguments on the following form

$$\begin{cases} \Delta(x_n + px_{n-\tau}) = a_n f(y_{n-l}) \\ \Delta y_n = b_n g(w_{n-m}), \\ \Delta w_n = \delta c_n h(x_{n-k}) \end{cases}$$

where the first equation of the the system is a neutral type difference equation. Firstly, the classification of nonoscillatory solutions of the considered system are presented. Next, we present the sufficient conditions for boundedness of a nonoscillatory solution. The obtained results are illustrated by an example.
Keywords Difference equation, neutral type, nonlinear system, nonoscillatory, bounded, unbounded solution
AMS Subject classification 39A10, 39A11, 39A12

1 Introduction

We consider a nonlinear three–dimensional difference system of the form

$$\begin{cases} \Delta(x_n + px_{n-\tau}) = a_n f(y_{n-l}), \\ \Delta y_n = b_n g(w_{n-m}), \\ \Delta w_n = \delta c_n h(x_{n-k}), \end{cases} \tag{1}$$

where Δ denotes the forward difference operator $\Delta z_n = z_{n+1} - z_n$ for any real sequence (z_n), $n \in \mathbb{N}_0 = \{n_0, n_0+1, \dots\}$, $n_0 = \max\{l, m, k, \tau\}$, $l, m, k, \tau \in \mathbb{N} = \{1, 2, \dots\}$ are given positive integers, p is a given real constant and $\delta = \pm 1$. Here sequences $(a_n), (b_n) : \mathbb{N}_0 \to \mathbb{R}_+ \cup \{0\}$, $(c_n) : \mathbb{N}_0 \to \mathbb{R}_+$, where $\mathbb{R}$, $\mathbb{R}_+$ denote the set of real numbers and the set of positive real numbers, respectively. Moreover

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \infty. \tag{2}$$

Assume that $f, g, h : \mathbb{R} \to \mathbb{R}$ are functions such that

$$uf(u) > 0, \quad ug(u) > 0, \quad uh(u) > 0 \quad \text{for } u \neq 0, \tag{3}$$

and there exists a positive constants M^*, M^{**} and M^{***} such that

$$\frac{f(u)}{u} \geq M^*, \frac{g(u)}{u} \geq M^{**}, \text{ and } \frac{h(u)}{u} \geq M^{***} \text{ for } u \neq 0. \tag{4}$$

Set $M = \min\{M^*, M^{**}, M^{***}\}$.

A solution $(x, y, w) = ((x_n), (y_n), (w_n))$ of the system (1) is called nonoscillatory if all its components are nonoscillatory (that is either eventually positive or eventually negative). A solution (x, y, w) of the system (1) is called bounded if all its components are bounded. Otherwise it is called unbounded.

The background for difference equations systems can be found in the well known monographs, see, for example: Agarwal, Bohner, Grace and O'Regan [2], Agarwal [1], Kocić and Ladas [4], and Kelly and Peterson [5].

In the last few years, great attention has been paid to the study of higher–order nonlinear difference equations. It is interesting to extend the known oscillation criteria for a larger class of higher order nonlinear difference equations.

Particularly, the oscillatory theory is considered usually for two–dimensional difference systems (see, for example: [6], [16], [22] and [23]).

Oscillatory results for three–dimensional system are investigated by Thandapani and Ponnammal in [21]. Results which are presented in this paper partially answered the open problems stated in the paper mentioned above.

The oscillatory and asymptotic properties of fourth–order nonlinear difference equations have been investigated, among many others, by: Došlá and Krejčová [3], Migda, Musielak, Popenda, Schmeidel and Szmanda [8–14], Smith and Taylor [17], and Arockiasamy, Dhanasekaran, Graef, Pandian and Thandapani [18–21], and references cited therein.

2 Some Basic Lemmas

Set

$$z_n = x_n + px_{n-\tau}. \tag{5}$$

The sequence (z_n) is called companion sequence of a sequence (x_n) relative to p.

We begin with some lemmas which will be useful for proving the main result of this paper. In 2005, Migda and Migda presented the following two results (see [7], Lemma 1 and Lemma 2).

Lemma 1. *Let (x_n), (p_n) be a real sequences and (z_n) be a sequence define by $z_n = x_n + p_n x_{n-\tau}$, for $n \geq \tau$. Assume that (x_n) is bounded, $\lim_{n\to\infty} z_n = l \in \mathbb{R}$, $\lim_{n\to\infty} p_n = \bar{p} \in \mathbb{R}$. If $|\bar{p}| \neq 1$, then (x_n) is convergent and $\lim_{n\to\infty} x_n = \frac{l}{1+\bar{p}}$.*

Notice that, the above Lemma holds obviously for $p_n \equiv p$.

Lemma 2. *Assume that* $x\colon \mathbb{N} \to \mathbb{R}$ *and*

$$|p| < 1.$$

If sequence (z_n) *defined by* (5) *is bounded, then sequence* (x_n) *is bounded too.*

Lemma 3. *Assume that* $p \geq 0$ *in* (5)*. If* $\lim_{n\to\infty} z_n = \infty$ *then* $\lim_{n\to\infty} x_n = \infty$.

Lemma 4. *Assume that condition* (3) *is satisfied. Let* (x, y, w) *be a solution of the system* (1) *such that sequence* (x_n) *is nonoscillatory. Then* (x, y, w) *is nonoscillatory and sequences* (y_n) *and* (w_n) *are monotonic for sufficiently large* n*.*

Proof. Sequence (x_n) is nonoscillatory it means that x_n is of the constant sign, for $n \in \mathbb{N}_0$. From the third equation of the system (1) and condition (3) we get that sequence (w_n) is monotonic. This implies that w_n is of the constant sign for enough large n. From the second equation of the system (1), sequence (y_n) is eventually monotonic. So, also sequence (y_n) is of the constant sign for enough large n.

Lemma 5. *Assume that*

$$p \in [0, 1) \tag{6}$$

and condition (3) *is satisfied. Let* (x, y, w) *be a solution of the system* (1) *and let sequence* (y_n) *(or* (w_n)*) be nonoscillatory. Then there exists limit of sequence* (x_n) *and exactly one of the following two cases hold:*

1. (x, y, w) *is nonoscillatory and sequences* (z_n), (y_n) *and* (w_n) *are monotonic for sufficiently large* n
2. $\lim_{n\to\infty} x_n = 0$.

Proof. Firstly, we assume that (y_n) is nonoscillatory. Since (y_n) is nonoscillatory then it is of the constant sign for large n. From the first equation of the system (1) and condition (3) we obtain that (z_n) is eventually monotonic. Hence, $\lim_{n\to\infty} z_n$ exists, finite or infinite. Let us consider three possible cases:

(i)

$$\lim_{n\to\infty} z_n = \pm\infty,$$

(ii)

$$\lim_{n\to\infty} z_n = L \neq 0,$$

(iii)

$$\lim_{n\to\infty} z_n = 0.$$

Case (i). Without loss of the generality we assume that $\lim_{n\to\infty} z_n = +\infty$. On the virtue of Lemma 3, we get $\lim_{n\to\infty} x_n = +\infty$. Hence, sequence (x_n) is of one sign for large n.
Case (ii). Since $\lim_{n\to\infty} z_n = L \neq 0$, sequence (z_n) is bounded. Lemma 2 now yields sequence (x_n) is also bounded. By Lemma 1, we get $\lim_{n\to\infty} x_n = \frac{L}{1+p} \neq 0$. It means that sequence (x_n) is of the one sign for large n.
Case (iii). Since $\lim_{n\to\infty} z_n = 0$, in a similar manner as in Case (ii), we get $\lim_{n\to\infty} x_n = 0$. Notice that, in this case sequence (x_n) is not need to be of the one sign for large n.

Finally, we assume that (w_n) is nonoscillatory. From the second equation of the system (1) and condition (3), we get that sequence (y_n) is eventually monotonic, so (y_n) is nonoscillatory. By similar arguments as before we get the thesis.

Lemma 6. *Assume that conditions* (2), (3), (4) *and* (6) *are satisfied. Let* (x, y, w) *be a nonoscillatory solution of the system* (1). *If*

$$\lim_{n\to\infty} x_n \in \mathbb{R},$$

then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} w_n = 0.$$

Proof. Let $\lim_{n\to\infty} x_n = c \in \mathbb{R}$. Hence, by Lemma 1, we have that $\lim_{n\to\infty} z_n = c(1+p)$ is finite too.

Because (x, y, w) is a nonoscillatory solution of (1) then, by Lemma 4, sequence (y_n) is monotonic. Hence limit of this sequence exists. Set

$$\lim_{n\to\infty} y_n = L^*.$$

For the sake of contradiction suppose that $L^* > 0$. (In the case $L^* < 0$ the proof is similar and hence omitted.) Since that, we have $y_n > 0$ for large n. Then there exists an integer $n_1 \geq n_0$ such that $y_{n-l} \geq \frac{L^*}{2}$, for $n \geq n_1$. By (4), there exists a positive constant M such that $f(y_{n-l}) \geq M y_{n-l} > 0$. Thus, from the first equation of the system (1), we have

$$\Delta z_n = a_n f(y_{n-l}) \geq M a_n y_{n-l} \geq M a_n \frac{L^*}{2} > 0.$$

Summing the above inequality from n_1 to $n-1$, we get

$$z_n \geq z_{n_1} + M \frac{L^*}{2} \sum_{i=n_1}^{n-1} a_i.$$

Taking $n \to \infty$, by (2), the right hand side of the above inequality tends to infinity, so the left hand side, too. This contradicts the fact that $\lim_{n\to\infty} z_n$ is finite. Therefore we get $\lim_{n\to\infty} y_n = 0$.

Analogously, using the second equation of the system (1), we obtain that $\lim_{n\to\infty} w_n = 0$. This complete the proof.

Lemma 7. *Assume that conditions* (2)–(4) *and* (6) *are satisfied, and* (x, y, w) *is a nonoscillatory solution of the system* (1). *Then exactly one of the following three cases holds:*

$$\begin{array}{ll} (I) & sgn\ x_n = sgn\ y_n = sgn\ w_n, \\ (II) & sgn\ x_n = sgn\ w_n \neq sgn\ y_n, \\ (III) & sgn\ x_n = sgn\ y_n \neq sgn\ w_n, \end{array}$$

for large n. *Moreover, if* $\delta = -1$ *in the system* (1) *then every nonoscillatory solution of* (1) *fulfills condition* (I) *or* (II), *if* $\delta = 1$ *then every nonoscillatory solution of* (1) *fulfills condition* (I) *or* (III).

Proof. Let (x, y, w) be a nonoscillatory solution of the system (1). Without loss of the generality assume that $x_n > 0$ for large n. This and assumption (6) imply that sequence (z_n) is eventually positive.

The proof falls naturally into two parts: $\delta = -1$ and $\delta = 1$.

Firstly, we assume that $\delta = -1$ in the system (1). Since (x_n) is nonoscillatory, on the virtue of Lemma 4, sequence (y_n) is eventually monotonic. Hence $y_n < 0$ or $y_n > 0$ for large n. Thus, by Lemma 5, we have that $\lim_{n\to\infty} x_n$ exists. There are two possible cases: $\lim_{n\to\infty} x_n = c^* \in \mathbb{R}$ or $\lim_{n\to\infty} x_n = \infty$. Since that, $\lim_{n\to\infty} z_n = c^*(1+p)$ or $\lim_{n\to\infty} z_n = \infty$, respectively.

Since (x_n) is nonoscillatory, by Lemma 4, we get that sequence (w_n) is eventually monotonic. Hence $w_n < 0$ or $w_n > 0$ for large n. For the sake of contradiction, suppose that $w_n < 0$ for large n. Then there exists n_2 such that $w_{n-m} < 0$ for $n \geq n_2$.

Since $x_n > 0$ for large n, from the third equation of the system (1), we get that sequence (w_n) is eventually decreasing. So $w_{n-m} < w_{n_2-m} < 0$ for $n \geq n_2$. By (4), we get $g(w_{n-m}) \leq M w_{n-m}$ for $n \geq n_2$. From this and the second equation of the system (1), we have

$$\Delta y_n \leq b_n M w_{n-m} < b_n M w_{n_2-m}$$

for $n \geq n_2$. Summing the above inequality from n_2 to $n-1$, we obtain

$$y_n < y_{n_2} + M w_{n_2-m} \sum_{i=n_2}^{n-1} b_i.$$

Taking n to infinity, by (2) and negativity of w_{n_2-m}, the right hand side of the above inequality tends to $-\infty$. So, the left hand side too. Hence $\lim_{n\to\infty} y_n = -\infty$. Then there exists an integer $n_3 \geq n_2$ such that $y_{n-l} < 0$ for $n \geq n_3$. From (4), we get $f(y_{n-l}) \leq M y_{n-l}$ for $n \geq n_3$. From the first equation of the system (1), we have

$$\Delta z_n \leq a_n M y_{n-l} < a_n M y_{n_3-l}$$

for $n \ge n_3$. Summing the above inequality from n_3 to $n-1$ and letting n to infinity, we get that $\lim_{n\to\infty} z_n = -\infty$. This contradicts positivity of the limit of sequence (z_n). On the virtue of this contradiction we exclude the case $w_n < 0$. So, we obtain that $w_n > 0$ for large n. It means that $sgn\ x_n = sgn\ w_n$ for enough large n. Therefore, for $\delta = -1$, case (I) or case (II) of the thesis of Lemma 7 is fulfilled.

Next, we assume that $\delta = 1$ in the system (1). From the third equation of the system (1) we get that sequence (w_n) is eventually increasing. Therefore $w_n < 0$ or $w_n > 0$ for large n. Let $w_n > 0$. From the second equation of the system (1) we have that sequence (y_n) is eventually of one sign. Hence $y_n < 0$ or $y_n > 0$ for large n. Suppose, for the sake of contradiction, that $y_n < 0$ eventually. This implies that sequence (z_n) is eventually nonincreasing. Since (z_n) is a positive and sequence (z_n) is a nonincreasing sequence, we have that $\lim_{n\to\infty} z_n$ is finite. By Lemma 2 and Lemma 1, it follows that $\lim_{n\to\infty} x_n$ is also finite. By Lemma 6, we have

$$\lim_{n\to\infty} w_n = 0.$$

This contradicts the fact that (w_n) is an eventually positive increasing sequence, and excludes the case that $y_n < 0$ for large n. Therefore also if $\delta = 1$ the thesis of Lemma 7 holds, because case (I) or case (III) of the thesis of Lemma 7 is satisfied. This completes the proof.

3 Main Results

Theorem 1. *Assume that conditions* (2)–(4) *are satisfied, and* $p \ge 0$. *Then every nonoscillatory solution* (x, y, w) *of the system* (1) *fulfilling condition* (I) *is unbounded.*

Proof. Let (x, y, w) be nonoscillatory solution of the system (1) for which condition (I) is satisfied. Without loss of the generality, assume that $x_n > 0$, $y_n > 0$ and $w_n > 0$ for large n, say $n \ge n_4$. From these and second equation of the system (1), we see that sequence (y_n) is eventually increasing. Summing the first equation of the system (1) from $n_5 = n_4 + l$ to $n-1$ we have

$$z_n = z_{n_5} + \sum_{i=n_5}^{n-1} a_i f(y_{i-l}), \text{ for } n \ge n_5.$$

Since $y_n \ge 0$, by the first equation of the system (1) and condition (3), we get that sequence (z_n) is monotonic. Then there exists $\lim_{n\to\infty} z_n$. Since $x_n > 0$ for sufficiently large n and by condition (3), we get $\lim_{n\to\infty} z_n = L^* \ge 0$. Therefore, there exists an integer $n_5 \ge n_0$ such that $z_n \ge \frac{L^*}{2}$, for $n \ge n_5$. From that, by positivity of y_n and by (4), we obtain

$$z_n \ge \frac{L^*}{2} + M \sum_{i=n_5}^{n-1} a_i y_{i-l}.$$

Since (y_n) is nondecreasing then

$$z_n \geq \frac{L^*}{2} + M y_{n_5 - l} \sum_{i=n_5}^{n-1} a_i .$$

Taking n to infty, using (2), we obtain that $\lim_{n\to\infty} z_n = \infty$. From the above and by Lemma 3, we see that $\lim_{n\to\infty} x_n = \infty$. Hence, every solution of the system (1) which fulfills (I) is unbounded.

Example 1. Let us consider the following system of difference equations, where $\delta = 1$,

$$\begin{cases} \Delta(x_n + 2x_{n-3}) = 58 y_{n-5}, \\ \Delta y_n = 54 w_{n-1}, \\ \Delta w_n = 18 x_{n-2}. \end{cases} \tag{7}$$

where $n \in \mathbb{N}$. All assumptions of the Theorem 1 hold. Hence this system has unbounded solution which satisfies condition (I). It easy to see that $(3^n, 3^{n+2}, 3^n)$ is such solution.

Example 2. Let us consider the following system of difference equations, where $\delta = -1$,

$$\begin{cases} \Delta(x_n + \frac{1}{2} x_{n-2}) = 19 y_{n-2}, \\ \Delta y_n = 2 \cdot 3^{2n-2} w_{n-2}, \\ \Delta w_n = -2 \cdot 3^{-2n+1} x_{n-2}. \end{cases} \tag{8}$$

where $n \in \mathbb{N}$. All assumptions of the Theorem 1 hold. Hence this system has unbounded solution which satisfies condition (I). It easy to see that $(3^n, 3^n, 3^{-n})$ is such solution.

Theorem 2. *Assume that conditions* (2)–(4) *and* (6) *are satisfied. Then every nonoscillatory solution* (x, y, w) *of the system* (1) *fulfilling condition* (II) *is bounded.*

Proof. Assume that (x, y, w) is nonoscillatory solution of the system (1) which satisfy condition (II). (Notice that, by Lemma 7, this system has such solution if and only if $\delta = -1$.) Without loss of the generality, we assume that $x_n > 0$, $y_n < 0$ and $w_n > 0$ for large n. Hence, from the first equation of the system (1), sequence (z_n) is decreasing. Taking it into account, by (6), we get (z_n) is positive too. We conclude that sequence (z_n) is bounded. On virtue of Lemma 2 and Lemma 1, we obtain that sequence (x_n) has also finite limit. So, the thesis holds.

The presented results improved and generalized those obtained by Schmeidel in 2010. Putting $p = 0$ in the system (1) we get the system considered in [15].

References

1. R. P. Agarwal, *Difference equations and inequalities. Theory, methods and applications*, Marcel Dekker, Inc., New York 1992.
2. R.P. Agarwal, M. Bohner, S.R. Grace and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing Corporation, New York, 2005.
3. Z. Došlá, J. Krejčová, *Oscillation of a class of the fourth–order nonlinear difference equations*, *Adv. Difference Equ.*, doi:10.1186/1687–1847–2012–99, (2012).
4. V.L. Kocić, G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1993.
5. W.G. Kelly, A.C. Peterson, *Difference equations*, Academic Press, Inc., Boston–San Diego 1991.
6. J.R. Graef, E. Thandapani, Oscillation of two–dimensional difference system, *Comput. Math. Appl.* 38, 7–8, 157–165 (1999).
7. M. Migda, J. Migda, *Asymptotic properties of solutions of second–order neutral difference equations*, *Nonlinear Anal.*, **63** e789—e799, (2005).
8. M. Migda, E. Schmeidel, *Asymptotic properties of fourth order nonlinear difference equations*, *Math. Comput. Modelling*, **39** 1203–1211, (2004).
9. M. Migda, A. Musielak, E. Schmeidel, *On a class of fourth order nonlinear difference equations*, *Adv. Difference Equ.*, **1** 23–36, (2004).
10. M. Migda, A. Musielak, E. Schmeidel, *Oscillatory of fourth order nonlinear difference equations with quasidifferences*,*Opuscula Math.*, **26(2)** 371–380, (2006).
11. J. Popenda, E. Schmeidel, *On the solution of fourth order difference equations*, *Rocky Mt. J. Math.*, **25(4)** 1485–1499, (1995).
12. E. Schmeidel, *Oscillation and nonoscillation theorems for fourth order difference equations*, *Rocky Mt. J.Math.*, **33(3)** 1083–1094, (2003).
13. E. Schmeidel, *Nonscillation and oscillation properties for fourth order nonlinear difference equations*, *New Progress in Difference Equations*, edited by B. Aulbach, S. Elaydi, G. Ladas, CRC, Boca Raton, FL, 531–538, (2004).
14. E. Schmeidel, B. Szmanda, *Oscillatory and asymptotic behavior of certain difference equation*, *Nonlinear Anal.*, **47** 4731–4742, (2001).
15. E. Schmeidel, *Boundedness of solutions of nonlinear three-dimensional difference systems with delays*, *Fasc. Math.*, **44**, 107–113, (2010).
16. Z. Szafrański, B. Szmanda, Oscillatory properties of solutions of some difference systems, *Rad. Mat.* 2, 205–214 (1990).
17. B. Smith, W.E. Taylor, *Oscillatory and asymptotic behavior of certain fourth order difference equations Rocky Mt. J. Math.* **16(2)** 403–406, (1986).
18. E. Thandapani, I.M. Arockiasamy, *On fourth order nonlinear oscillations of difference equations Comput. Math. Appl.* **42** 357–368, (2001).
19. E. Thandapani, J.R. Graef, *Oscillatory and asymptotic behavior of fourth order nonlinear delay difference equations Fasc. Math.* **31** 23–36, (2001).
20. E. Thandapani, S. Pandian, R. Dhanasekaran, J. Graef, *Asymptotic results for a class of fourth order quasilinear difference equations J. Differ. Equ. Appl.* **13(12)** 1085–1103, (2007).
21. E. Thandapani, B. Ponnammal, Oscillatory properties of solutions of three dimensional difference systems, *Math. Comput. Modelling* 42, 5–6, 641–650 (2005).
22. E. Thandapani, B. Ponnammal, On the oscillation of a nonlinear two–dimensional difference system, *Tamkang J. Math.* 32, 3, 201–209 (2001).

23. E. Thandapani, B. Ponnammal, Oscillatory and asymptotic behavior of solutions of nonlinear two–dimensional difference systems, *Math. Sci. Res. Hot–Line* 4, 1, 1–18 (2000).

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On Discretization of Polynomials Corresponding to Symmetric and Antisymmetric Functions in Four Variables

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Abstract. The paper describes the symmetric and antisymmetric exponential functions of four variables, based on the permutation group S_4. We derive the explicit formulas for corresponding families of the orthogonal polynomials and some of their properties, namely the orthogonality, both the continuous one and the discrete one on the lattice. A general formula for orthogonal polynomials in n variables connected with symmetric exponential functions is given.

1 Introduction

This paper aims at a short presentation of a direct approach to the functions of four variables defined on Euclidean space, possessing the symmetry/antisymmetry of the permutation group S_n. Such functions form a family of special functions, called the symmetric/antisymmetric exponential functions. They were introduced in [6] in general for any integer n.

In Section 2 the symmetric exponential functions are defined and their basic properties including product decompositions are shown. Section 4 contains the description of antisymmetric exponential functions and their basic properties. We construct also sets of orthogonal polynomials, which are defined by a certain substitution rule, similarly to some cases of the polynomials in [3]. The families of these orthogonal polynomials correspond to the symmetric/antisymmetric exponential functions. The explicit formulas for these polynomials are deduced and the continuous orthogonality of these polynomials as well as their discrete orthogonality are explicitly formulated.

The symmetry/antisymmetry with respect to permutations can be useful in description of dynamical systems possessing such a symmetry, as well as in problems of quantum information processing.

2 Four dimensional symmetric exponential functions

2.1 Definitions, symmetries and general properties

The four dimensional symmetric exponential functions $E^+_{(\lambda,\mu,\nu,\xi)}\colon \mathbb{R}^4 \to \mathbb{C}$ are, for λ, μ, $\nu, \xi \in \mathbb{R}$, given by the explicit formula [6]

$$\begin{aligned}
E^+_{(\lambda,\mu,\nu,\xi)}(x,y,z,t) &= \begin{vmatrix} e^{2\pi i\lambda x} & e^{2\pi i\lambda y} & e^{2\pi i\lambda z} & e^{2\pi i\lambda t} \\ e^{2\pi i\mu x} & e^{2\pi i\mu y} & e^{2\pi i\mu z} & e^{2\pi i\mu t} \\ e^{2\pi i\nu x} & e^{2\pi i\nu y} & e^{2\pi i\nu z} & e^{2\pi i\nu t} \\ e^{2\pi i\xi x} & e^{2\pi i\xi y} & e^{2\pi i\xi z} & e^{2\pi i\xi t} \end{vmatrix}^+ \\
&= e^{2\pi i(\lambda x+\mu y+\nu z+\xi t)} + e^{2\pi i(\lambda x+\mu z+\nu t+\xi y)} + e^{2\pi i(\lambda x+\mu t+\nu y+\xi z)} \\
&+ e^{2\pi i(\lambda x+\mu t+\nu z+\xi y)} + e^{2\pi i(\lambda x+\mu y+\nu t+\xi z)} + e^{2\pi i(\lambda x+\mu z+\nu y+\xi t)} \\
&+ e^{2\pi i(\lambda y+\mu x+\nu z+\xi t)} + e^{2\pi i(\lambda y+\mu t+\nu x+\xi z)} + e^{2\pi i(\lambda y+\mu z+\nu t+\xi x)} \\
&+ e^{2\pi i(\lambda y+\mu t+\nu z+\xi x)} + e^{2\pi i(\lambda y+\mu x+\nu t+\xi z)} + e^{2\pi i(\lambda y+\mu z+\nu x+\xi t)} \\
&+ e^{2\pi i(\lambda z+\mu x+\nu y+\xi t)} + e^{2\pi i(\lambda z+\mu y+\nu t+\xi x)} + e^{2\pi i(\lambda z+\mu t+\nu x+\xi y)} \\
&+ e^{2\pi i(\lambda z+\mu t+\nu y+\xi x)} + e^{2\pi i(\lambda z+\mu x+\nu t+\xi y)} + e^{2\pi i(\lambda z+\mu y+\nu x+\xi t)} \\
&+ e^{2\pi i(\lambda t+\mu x+\nu y+\xi z)} + e^{2\pi i(\lambda t+\mu z+\nu x+\xi y)} + e^{2\pi i(\lambda t+\mu y+\nu z+\xi x)} \\
&+ e^{2\pi i(\lambda t+\mu z+\nu y+\xi x)} + e^{2\pi i(\lambda t+\mu x+\nu z+\xi y)} + e^{2\pi i(\lambda t+\mu y+\nu x+\xi z)}.
\end{aligned} \tag{1}$$

The functions $E^+_{(\lambda,\mu,\nu,\xi)}$ are symmetric with respect to all permutations of the variables (x,y,z,t). Similar invariance, with respect to the twenty four permutations of the labels (λ,μ,ν,ξ), holds when the functions $E^+_{(\lambda,\mu,\nu,\xi)}$ are evaluated at a fixed point (x,y,z,t).

Therefore only $E^+_{(\lambda,\mu,\nu,\xi)}$ with the so called dominant quadruple (λ,μ,ν,ξ) can be considered, that is such quadruples (λ,μ,ν,ξ) where $\lambda \geq \mu \geq \nu \geq \xi$. The functions $E^+_{(k,l,m,n)}$ with $k,l,m,n \in \mathbb{Z}$ satisfy the invariance with respect to integer shifts,

$$E^+_{(k,l,m,n)}(x+r,y+s,z+u,t+w) = E^+_{(k,l,m,n)}(x,y,z,t), \quad r,\ s,\ u,w \in \mathbb{Z} \tag{2}$$

and moreover for $a \in \mathbb{R}$ we have

$$E^+_{(k,l,m,n)}(x+a,y+a,z+a,t+a) = e^{2\pi i(k+l+m+n)a} E^+_{(k,l,m,n)}(x,y,z,t). \tag{3}$$

The properties (2) and (3) let us consider the function $E^+_{(k,l,m,n)}$, where $k,l,m,n \in \mathbb{Z}$, on the closure of *fundamental domain* $F(S_4^{\mathrm{aff}})$ only [6]. The fundamental domain $F(S_4^{\mathrm{aff}})$ is defined explicitly as the set

$$F(S_4^{\mathrm{aff}}) = \{(x,y,z,t) \in (0,1)\times(0,1)\times(0,1)\times(0,1) \mid x>y>z>t\}\,. \tag{4}$$

2.2 Product decompositions

The product of two $4D$ symmetric functions evaluated at the same point (x, y, z, t) can be decomposed into the sum of the symmetric functions with suitable indices. The decomposition formula has the following explicit form:

$$
\begin{aligned}
&E^+_{(\lambda,\mu,\nu,\xi)}(x,y,z,t)E^+_{(a,b,c,d)}(x,y,z,t)\\
&\quad= E^+_{(\lambda+a,\mu+b,\nu+c,\xi+d)}(x,y,z,t) + E^+_{(\lambda+d,\mu+a,\nu+b,\xi+c)}(x,y,z,t)\\
&\quad+ E^+_{(\lambda+c,\mu+d,\nu+a,\xi+b)}(x,y,z,t) + E^+_{(\lambda+b,\mu+c,\nu+d,\xi+a)}(x,y,z,t)\\
&\quad+ E^+_{(\lambda+a,\mu+d,\nu+b,\xi+c)}(x,y,z,t) + E^+_{(\lambda+a,\mu+c,\nu+d,\xi+b)}(x,y,z,t)\\
&\quad+ E^+_{(\lambda+a,\mu+d,\nu+c,\xi+b)}(x,y,z,t) + E^+_{(\lambda+a,\mu+b,\nu+d,\xi+c)}(x,y,z,t)\\
&\quad+ E^+_{(\lambda+a,\mu+c,\nu+b,\xi+d)}(x,y,z,t) + E^+_{(\xi+a,\lambda+b,\mu+c,\nu+d)}(x,y,z,t)\\
&\quad+ E^+_{(\xi+a,\lambda+d,\mu+b,\nu+c)}(x,y,z,t) + E^+_{(\xi+a,\lambda+c,\mu+d,\nu+b)}(x,y,z,t)\\
&\quad+ E^+_{(\xi+a,\lambda+c,\mu+b,\nu+d)}(x,y,z,t) + E^+_{(\xi+a,\lambda+d,\mu+c,\nu+b)}(x,y,z,t)\\
&\quad+ E^+_{(\nu+a,\xi+b,\lambda+c,\mu+d)}(x,y,z,t) + E^+_{(\nu+a,\xi+d,\lambda+b,\mu+c)}(x,y,z,t)\\
&\quad+ E^+_{(\nu+a,\xi+c,\lambda+d,\mu+b)}(x,y,z,t) + E^+_{(\nu+a,\xi+c,\lambda+b,\mu+d)}(x,y,z,t)\\
&\quad+ E^+_{(\nu+a,\xi+d,\lambda+c,\mu+b)}(x,y,z,t) + E^+_{(\mu+a,\nu+b,\xi+c,\lambda+d)}(x,y,z,t)\\
&\quad+ E^+_{(\mu+a,\nu+d,\xi+b,\lambda+c)}(x,y,z,t) + E^+_{(\mu+a,\nu+c,\xi+d,\lambda+b)}(x,y,z,t)\\
&\quad+ E^+_{(\mu+a,\nu+c,\xi+b,\lambda+d)}(x,y,z,t) + E^+_{(\mu+a,\nu+d,\xi+c,\lambda+b)}(x,y,z,t).
\end{aligned}
\tag{5}
$$

Analogously, we obtain a product-to-sum decomposition formula for one function $E^+_{\lambda,\mu,\nu,\xi}$ evaluated at two different points:

$$
\begin{aligned}
&E^+_{(\lambda,\mu,\nu,\xi)}(x,y,z,t)E^+_{(\lambda,\mu,\nu,\xi)}(x',y',z',t') = \\
=&E^+_{(\lambda,\mu,\nu,\xi)}(x+x',y+y',z+z',t+t') + E^+_{(\lambda,\mu,\nu,\xi)}(x+t',y+x',z+y',t+z') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(x+z',y+t',z+x',t+y') + E^+_{(\lambda,\mu,\nu,\xi)}(x+y',y+z',z+t',t+x') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(x+x',y+t',z+y',t+z') + E^+_{(\lambda,\mu,\nu,\xi)}(x+x',y+z',z+t',t+y') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(x+x',y+t',z+z',t+y') + E^+_{(\lambda,\mu,\nu,\xi)}(x+x',y+y',z+t',t+z') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(x+x',y+z',z+y',t+t') + E^+_{(\lambda,\mu,\nu,\xi)}(t+x',x+y',y+z',z+t') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(t+x',x+t',y+y',z+z') + E^+_{(\lambda,\mu,\nu,\xi)}(t+x',x+z',y+t',z+y') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(t+x',x+z',y+y',z+t') + E^+_{(\lambda,\mu,\nu,\xi)}(t+x',x+t',y+z',z+y') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(z+x',t+y',x+z',y+t') + E^+_{(\lambda,\mu,\nu,\xi)}(z+x',t+t',x+y',y+z') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(z+x',t+z',x+t',y+y') + E^+_{(\lambda,\mu,\nu,\xi)}(z+x',t+z',x+y',y+t') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(z+x',t+t',x+z',y+y') + E^+_{(\lambda,\mu,\nu,\xi)}(y+x',z+y',t+z',x+t') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(y+x',z+t',t+y',x+z') + E^+_{(\lambda,\mu,\nu,\xi)}(y+x',z+z',t+t',x+y') \\
+&E^+_{(\lambda,\mu,\nu,\xi)}(y+x',z+z',t+y',x+t') + E^+_{(\lambda,\mu,\nu,\xi)}(y+x',z+t',t+z',x+y').
\end{aligned}
$$

2.3 Orthogonal polynomials corresponding to symmetric exponential functions

The set of continuous and discrete orthogonal polynomials can be built using the functions $E^+_{(k,l,m,n)}$ with dominant quadruples (k,l,m,n) and $k,l,m,n \in \mathbb{Z}^{\geq 0}$. Introducing the following substitutions for the lowest four symmetric exponential functions,

$$X = E^+_{(1,0,0,0)}, \quad Y = E^+_{(1,1,0,0)}, \quad Z = E^+_{(1,1,1,0)}, \quad T = E^+_{(1,1,1,1)}, \tag{6}$$

four-variable polynomials $P^+_{(k,l,m,n)}(X,Y,Z,T)$ labeled by the dominant quadruples (k,l,m,n), with $k \geq l \geq m \geq n$, are defined by the equation

$$P^+_{(k,l,m,n)}(X(x,y,z,t),Y(x,y,z,t),Z(x,y,z,t),T(x,y,z,t)) = E^+_{(k,l,m,n)}(x,y,z,t). \tag{7}$$

First polynomials $P^+_{(k,l,m,n)}(X,Y,Z,T)$ are given in the following table:

$$
\begin{array}{ll}
P^+_{(0,0,0,0)} = 24 & P^+_{(2,2,0,0)} = \dfrac{9Y^2 + 12T - 8XZ}{36} \\
P^+_{(1,0,0,0)} = X & P^+_{(2,2,1,0)} = \dfrac{1}{24}X(2Z - T) \\
P^+_{(1,1,0,0)} = Y & P^+_{(2,2,1,1)} = \dfrac{1}{24}YT \\
P^+_{(1,1,1,0)} = Z & P^+_{(2,2,2,0)} = \dfrac{1}{5}\left(Z^2 - \dfrac{19}{24}YT\right) \\
P^+_{(1,1,1,1)} = T & P^+_{(2,2,2,1)} = \dfrac{1}{24}ZT \\
P^+_{(2,0,0,0)} = \dfrac{1}{6}\left(X^2 - 18Y\right) & P^+_{(2,2,2,2)} = \dfrac{1}{24}T^2 \\
P^+_{(2,1,0,0)} = \dfrac{1}{12}(XY - 12Z) & P^+_{(3,0,0,0)} = \dfrac{1}{36}\left(X^3 - 27XY + 108Z\right) \\
P^+_{(2,1,1,0)} = \dfrac{1}{18}(XZ - 6T) & P^+_{(3,1,0,0)} = \dfrac{1}{72}\left(X^2Y - 4XZ - 18Y^2 + 24T\right) \\
P^+_{(2,1,1,1)} = \dfrac{1}{24}XT & P^+_{(3,1,1,0)} = \dfrac{1}{216}\left(2X^2Z - 36Y - 3XT\right)
\end{array}
$$

$$
\begin{aligned}
P^+_{(3,1,1,1)} &= \frac{1}{144}T(X^2-18Y)\\
P^+_{(3,2,0,0)} &= \frac{144Y-72XZ+30XT-108YZ+9XY^2-8X^2Z}{432}\\
P^+_{(3,2,1,0)} &= \frac{1}{288}\left(12YT-X^2T-24Z^2+2XYZ\right)\\
P^+_{(3,2,1,1)} &= \frac{1}{288}T(XY-12Z)\\
P^+_{(3,2,2,0)} &= \frac{1}{1440}\left(16XZ^2-12ZT-13XYT\right)
\end{aligned}
$$

$$
\begin{aligned}
P^+_{(3,2,2,1)} &= \frac{1}{432}T(XZ-6T)\\
P^+_{(3,2,2,2)} &= \frac{1}{24^2}XT^2\\
P^+_{(3,3,0,0)} &= \frac{18Y^3+3X^2T+72Z^2-48XZ+24XT-12YT-22XYZ}{288}\\
P^+_{(3,3,1,0)} &= \frac{1}{864}(18Y^2Z-16XZ^2-3XYT+60ZT)\\
P^+_{(3,3,1,1)} &= \frac{1}{864}Z(12T+9Y^2-8XZ)\\
P^+_{(3,3,2,0)} &= \frac{1}{4320}\left(128XZ^2-60XZT-63Y^2Z-84ZT+144T^2\right)\\
P^+_{(3,3,2,1)} &= \frac{1}{24^2}XT(2Z-T)
\end{aligned}
$$

Explicit formula for $P^+_{(k,l,m,n)}$ Note that in view of the decomposition rule (5) for any admissible k,l,m,n, the following relations hold:

$$P^+_{(k,l,m,n)}P^+_{(0,0,0,0)} = 24P^+_{(k,l,m,n)},$$

$$P^+_{(k-1,l-1,m-1,n-1)}P^+_{(1,1,1,1)} = 24P^+_{(k,l,m,n)}.$$

Therefore

$$P^+_{(k,l,m,n)} = \frac{T}{24}P^+_{(k-1,l-1,m-1,n-1)}.$$

Iterating the last relation we obtain

$$P^+_{(k,l,m,n)} = \left(\frac{T}{24}\right)^n P^+_{(k-n,l-n,m-n,0)}. \tag{8}$$

Polynomials $P^+_{(k,l,m,n)}$ fulfil the recurrence relation

$$P^+_{(k+4,l,m,0)} - \frac{X}{6}P^+_{(k+3,l,m,0)} + \frac{Y}{4}P^+_{(k+2,l,m,0)} - \frac{Z}{6}P^+_{(k+1,l,m,0)} + \frac{T}{24}P^+_{(k,l,m,0)} = 0 \tag{9}$$

which follows from the decomposition rule (5).

General solution of the difference equation (9) has the following form:

$$P^{+}_{(k,l,m,0)} = c_1^{(l,m)}\lambda_1^k + c_2^{(l,m)}\lambda_2^k + c_3^{(l,m)}\lambda_3^k + c_4^{(l,m)}\lambda_4^k, \tag{10}$$

where $\lambda_j, j = 1,2,3,4$ are roots of the characteristic equation

$$\lambda^4 - \frac{X}{6}\lambda^3 + \frac{Y}{4}\lambda^2 - \frac{Z}{6}\lambda + \frac{T}{24} = 0$$

corresponding to the equation (9). The roots λ_j can be expressed as

$$\lambda_1 = \frac{1}{264}\,C_1 + \frac{1}{264}\,C_2 + \frac{1}{264}\,C_3 + \frac{1}{24}X$$

$$\lambda_2 = \frac{1}{264}\,C_1 - \frac{1}{264}\,C_2 - \frac{1}{264}\,C_3 + \frac{1}{24}X$$

$$\lambda_3 = -\frac{1}{264}\,C_1 + \frac{1}{264}\,C_2 - \frac{1}{264}\,C_3 + \frac{1}{24}X$$

$$\lambda_4 = -\frac{1}{264}\,C_1 - \frac{1}{264}\,C_2 + \frac{1}{264}\,C_3 + \frac{1}{24}X,$$

where

$$C_1 = \sqrt{-22\sqrt[3]{A} + 13799808B - 2904\,Y + 121\,X^2}$$

$$C_2 = \sqrt{11\,\sqrt[3]{A} - 6899904B - 2904\,Y + 121\,X^2 - 8712\,i\sqrt{3}\left(\frac{1}{792}\,\sqrt[3]{A} + 792B\right)}$$

$$C_3 = \sqrt{11\,\sqrt[3]{A} - 6899904B - 2904\,Y + 121\,X^2 + 8712\,i\sqrt{3}\left(\frac{1}{792}\,\sqrt[3]{A} + 792B\right)}$$

$$\begin{aligned} A = &- 287496Y^3 - 2038608YXZ + 108900ZX^3 + 6899904TY \\ &- 287496TX^2 - 6899904Z^2 + 396\sqrt{W} \end{aligned}$$

$$\begin{aligned} W = &\ 14374800\,Y^4XZ + 17058096\,YX^3ZT + 303595776\,Z^4 \\ &- 37949472\,Y^4T + 202397184\,Y^2T^2 + 34211232\,X^3Z^3 \\ &- 3606768\,Y^2X^2Z^2 - 215569728\,X^2Z^2T + 441593856\,XZT^2 \\ &- 399300\,Y^3ZX^3 + 1054152Y^3TX^2 - 2831400\,YX^4Z^2 \\ &- 399300\,ZX^5T - 25299648\,T^2YX^2 - 607191552\,TYZ^2 \\ &- 269862912\,T^3 + 179397504\,YXZ^3 - 68999040\,Y^2XZT \\ &+ 25299648\,Y^3Z^2 + 75625\,Z^2X^6 + 527076\,T^2X^4 \end{aligned}$$

$$B=\left(-\frac{1}{144}Y^2+\frac{1}{33}XZ-\frac{1}{18}T\right)A^{-1/3}.$$

Since polynomials $P^+_{(k,l,m,n)}$ are symmetric with respect to all permutations of indices, we have $P^+_{(k,l,m,0)}=P^+_{(l,m,k,0)}=P^+_{(m,k,l,0)}$ and this allows us to write the system of linear equations

$$\begin{aligned}&P^+_{(0,l,m,0)}=c_1^{(l,m)}+c_2^{(l,m)}+c_3^{(l,m)}+c_4^{(l,m)}\\=&P^+_{(m,0,l,0)}=c_1^{(0,l)}\lambda_1^m+c_2^{(0,l)}\lambda_2^m+c_3^{(0,l)}\lambda_3^m+c_4^{(0,l)}\lambda_4^m,\end{aligned}$$

$$\begin{aligned}&P^+_{(1,l,m,0)}=c_1^{(l,m)}\lambda_1+c_2^{(l,m)}\lambda_2+c_3^{(l,m)}\lambda_3+c_4^{(l,m)}\lambda_4\\=&P^+_{(m,1,l,0)}=c_1^{(1,l)}\lambda_1^m+c_2^{(1,l)}\lambda_2^m+c_3^{(1,l)}\lambda_3^m+c_4^{(1,l)}\lambda_4^m,\end{aligned}$$

$$\begin{aligned}&P^+_{(2,l,m,0)}=c_1^{(l,m)}\lambda_1^2+c_2^{(l,m)}\lambda_2^2+c_3^{(l,m)}\lambda_3^2+c_4^{(l,m)}\lambda_4^2\\=&P^+_{(m,2,l,0)}=c_1^{(2,l)}\lambda_1^m+c_2^{(2,l)}\lambda_2^m+c_3^{(2,l)}\lambda_3^m+c_4^{(2,l)}\lambda_4^m,\end{aligned}$$

$$\begin{aligned}&P^+_{(3,l,m,0)}=c_1^{(l,m)}\lambda_1^3+c_2^{(l,m)}\lambda_2^3+c_3^{(l,m)}\lambda_3^3+c_4^{(l,m)}\lambda_4^3\\=&P^+_{(m,3,l,0)}=c_1^{(3,l)}\lambda_1^m+c_2^{(3,l)}\lambda_2^m+c_3^{(3,l)}\lambda_3^m+c_4^{(3,l)}\lambda_4^m,\end{aligned}$$

and

$$\begin{aligned}P^+_{(0,l,m,0)}&=P^+_{(l,m,0,0)}=c_1^{(m,0)}\lambda_1^l+c_2^{(m,0)}\lambda_2^l+c_3^{(m,0)}\lambda_3^l+c_4^{(m,0)}\lambda_4^l,\\P^+_{(1,l,m,0)}&=P^+_{(l,m,1,0)}=c_1^{(m,1)}\lambda_1^l+c_2^{(m,1)}\lambda_2^l+c_3^{(m,1)}\lambda_3^l+c_4^{(m,1)}\lambda_4^l,\\P^+_{(2,l,m,0)}&=P^+_{(l,m,2,0)}=c_1^{(m,2)}\lambda_1^l+c_2^{(m,2)}\lambda_2^l+c_3^{(m,2)}\lambda_3^l+c_4^{(m,2)}\lambda_4^l,\\P^+_{(3,l,m,0)}&=P^+_{(l,m,3,0)}=c_1^{(m,3)}\lambda_1^l+c_2^{(m,3)}\lambda_2^l+c_3^{(m,3)}\lambda_3^l+c_4^{(m,3)}\lambda_4^l.\end{aligned}$$

The coefficients $c_j^{(l,m)}$ in (10) can be found from above systems, where the coefficients $c_j^{(0,l)}$, $c_j^{(1,l)}$, $c_j^{(2,l)}$, $c_j^{(3,l)}$ and $c_j^{(m,0)}$, $c_j^{(m,1)}$, $c_j^{(m,2)}$, $c_j^{(m,3)}$ are obtained from the initial conditions.

For $P^+_{(k,l,m,0)}$ we finally get

$$\begin{aligned}P^+_{(k,l,m,0)}=&(\lambda_2^l\lambda_3^m+\lambda_2^l\lambda_4^m+\lambda_3^l\lambda_4^m+\lambda_2^m\lambda_3^l+\lambda_2^m\lambda_4^l+\lambda_3^m\lambda_4^l)\lambda_1^k\\&+(\lambda_1^l\lambda_3^m+\lambda_1^l\lambda_4^m+\lambda_3^l\lambda_4^m+\lambda_1^m\lambda_3^l+\lambda_1^m\lambda_4^l+\lambda_3^m\lambda_4^l)\lambda_2^k\\&+(\lambda_1^l\lambda_2^m+\lambda_1^l\lambda_4^m+\lambda_2^l\lambda_4^m+\lambda_1^m\lambda_2^l+\lambda_1^m\lambda_4^l+\lambda_2^m\lambda_4^l)\lambda_3^k\\&+(\lambda_1^l\lambda_2^m+\lambda_1^l\lambda_3^m+\lambda_2^l\lambda_3^m+\lambda_1^m\lambda_2^l+\lambda_1^m\lambda_3^l+\lambda_2^m\lambda_3^l)\lambda_4^k.\end{aligned}$$

Using the relation (8) we can write the explicit formula for the polynomials $P^+_{(k,l,m,n)}$ in the form

$$P^+_{(k,l,m,n)} = \left(\frac{T}{24}\right)^n \times$$
$$\times\Big[(\lambda_2^{l-n}\lambda_3^{m-n}+\lambda_2^{l-n}\lambda_4^{m-n}+\lambda_3^{l-n}\lambda_4^{m-n}+\lambda_2^{m-n}\lambda_3^{l-n}+\lambda_2^{m-n}\lambda_4^{l-n}+\lambda_3^{m-n}\lambda_4^{l-n})\lambda_1^{k-n}$$
$$+(\lambda_1^{l-n}\lambda_3^{m-n}+\lambda_1^{l-n}\lambda_4^{m-n}+\lambda_3^{l-n}\lambda_4^{m-n}+\lambda_1^{m-n}\lambda_3^{l-n}+\lambda_1^{m-n}\lambda_4^{l-n}+\lambda_3^{m-n}\lambda_4^{l-n})\lambda_2^{k-n}$$
$$+(\lambda_1^{l-n}\lambda_2^{m-n}+\lambda_1^{l-n}\lambda_4^{m-n}+\lambda_2^{l-n}\lambda_4^{m-n}+\lambda_1^{m-n}\lambda_2^{l-n}+\lambda_1^{m-n}\lambda_4^{l-n}+\lambda_2^{m-n}\lambda_4^{l-n})\lambda_3^{k-n}$$
$$+(\lambda_1^{l-n}\lambda_2^{m-n}+\lambda_1^{l-n}\lambda_3^{m-n}+\lambda_2^{l-n}\lambda_3^{m-n}+\lambda_1^{m-n}\lambda_2^{l}+\lambda_1^{m-n}\lambda_3^{l-n}+\lambda_2^{m-n}\lambda_3^{l-n})\lambda_4^{k-n}\Big].$$

Continuous orthogonality of $P^+_{(k,l,m,n)}$ The Jacobi determinant of the mapping (6) is equal to

$$J(x,y,z,t) = 55296\,\pi^4 e^{2\pi i(x+y+z+t)}\left(e^{2\pi ix}-e^{2\pi iy}\right)\left(e^{2\pi ix}-e^{2\pi iz}\right)$$
$$\times\left(e^{2\pi ix}-e^{2\pi it}\right)\left(e^{2\pi iy}-e^{2\pi iz}\right)\left(e^{2\pi iy}-e^{2\pi it}\right)\left(e^{2\pi iz}-e^{2\pi it}\right).$$

Changing the variables, according to (6), the Jacobian J may be expressed in variables X,Y,Z,T.

The polynomials $P^+_{(k,l,m,n)}$ are mutually orthogonal on the F_4 — transformed fundamental region $F(S_4^{\mathrm{aff}})$ via mapping (6), i.e. scalar product of polynomials $P^+_{(k,l,m,n)}$ and $P^+_{(k',l',m',n')}$ is then given by the formula

$$\begin{aligned}\left(P^+_{(k,l,m,n)},P^+_{(k',l',m',n')}\right) &= \int_{F_4} P^+_{(k,l,m,n}(X,Y,Z,T)\overline{P^+_{(k',l',m',n')}(X,Y,Z,T)}\times\\ &\times\frac{1}{J(X,Y,Z,T)}\mathrm{d}X\mathrm{d}Y\mathrm{d}Z\mathrm{d}T \qquad (11)\\ &= G^+_{klmn}\delta_{kk'}\delta_{ll'}\delta_{mm'}\delta_{nn'},\end{aligned}$$

where

$$G^+_{klmn} = \begin{cases} 24, & \text{if } S_1,\\ 6, & \text{if } S_2 \text{ and not } S_1,\\ 4, & \text{if } S_3 \text{ and not } S_2 \text{ and not } S_1,\\ 2, & \text{if } S_4 \text{ and not } S_3 \text{ and not } S_2 \text{ and not } S_1,\\ 1, & \text{otherwise}\end{cases}$$

and S_1,S_2,S_3,S_4 are the following conditions:
$S_1: k=l=m=n$
$S_2: k=l=m \quad\text{or}\quad k=l=n \quad\text{or}\quad k=m=n \quad\text{or}\quad l=m=n$
$S_3: (k=l\ \wedge\ m=n) \quad\text{or}\quad (k=m\ \wedge\ l=n) \quad\text{or}\quad (k=n\ \wedge\ l=m)$
$S_4: k=l \quad\text{or}\quad k=m \quad\text{or}\quad k=n \quad\text{or}\quad l=m \quad\text{or}\quad l=n \quad\text{or}\quad m=n.$

Discrete orthogonality of $P^+_{(k,l,m,n)}$ Given an arbitrary natural number N, the discrete calculus of exponential $E^+_{(k,l,m,n)}$ functions is performed on the finite grid L^+_N. The grid L^+_N, with its points contained inside the closure of the fundamental domain $F(S_4^{\mathrm{aff}})$, is given explicitly as

$$L^+_N = \left\{ \left(\frac{r}{N}, \frac{s}{N}, \frac{u}{N}, \frac{w}{N} \right) \,\middle|\, r \geq s \geq u \geq w;\ r,s,u,w = 0,1,2,...,N-1 \right\}. \quad (12)$$

The positive integer N fixes the density of the grid inside $\overline{F(S_4^{\mathrm{aff}})}$ and the grid L^+_N contains $\frac{1}{24}N(N+1)(N+2)(N+3)$ points. The discrete set $\mathcal{L}^+_N$ is the grid L^+_N transformed via the mapping (6) and is given as

$$\mathcal{L}^+_N = \{ (X(x_r,y_s,z_u,t_w), Y(x_r,y_s,z_u,t_w), Z(x_r,y_s,z_u,t_w), T(x_r,y_s,z_u,t_w)) \mid (x_r,y_s,z_u,t_w) \in L^+_N \}.$$

Labeling each point

$$(X_r,Y_s,Z_u,T_w) \equiv (X(x_r,y_s,z_u,t_w), Y(x_r,y_s,z_t), Z(x_r,y_s,z_u,t_w), T(x_r,y_s,z_u,t_w))$$

from $\mathcal{L}^+_N$ by the corresponding (r,s,u,w) from (12), the discrete orthogonality of the polynomials $P^+_{(k,l,m,n)}, P^+_{(k',l',m',n')}$, which holds for any (k,l,m,n), (k',l',m',n'), is of the explicit form

$$\sum_{(X_r,Y_s,Z_u,T_w)\in\mathcal{L}^+_N} (G^+_{rsuw})^{-1} P^+_{(k,l,m,n)}(X_r,Y_s,Z_u,T_w) \overline{P^+_{(k',l',m',n')}(X_r,Y_s,Z_u,T_w)} = G^+_{klmn} N^4 \delta_{kk'}\delta_{ll'}\delta_{mm'}\delta_{nn'}.$$

2.4 Symmetric discrete Fourier transform

The symmetric discrete Fourier transform of a discrete complex function f is given by

$$\beta^+_{klmn} = \frac{1}{G^+_{klmn}N^4} \times \sum_{(X_r,Y_s,Z_u,T_w)\in\mathcal{L}^+_N} (G^+_{rsuw})^{-1} f(X_r,Y_s,Z_u,T_w) \overline{P^+_{(k,l,m,n)}(X_r,Y_s,Z_u,T_w)}, \quad (13)$$

where $k \geq l \geq m \geq n,\ k,l,m,n = 0,1,2,\ldots,N-1$. Discrete orthogonality relations for polynomials $P^+_{(k,l,m,n)}$ immediately give the inverse transform of the coefficients β^+_{klm}:

$$f(X_r,Y_s,Z_u,T_w) = \sum_{\substack{k,l,m,n=0\\ k\geq l\geq m\geq n}}^{N-1} \beta^+_{klmn} P^+_{(k,l,m,n)}(X_r,Y_s,Z_u,T_w). \quad (14)$$

3 General formula for polynomials $P^+_{(k_1,k_2,\ldots,k_n)}$

Examining the cases with $n \le 4$ ([1],[2]) we can notice that the general recurrence formula for polynomials $P^+_{(k_1,k_2,\ldots,k_n)}(X_1, X_2, \ldots, X_n)$ corresponding to symmetric exponential functions $E^+_{(k_1,k_2,\ldots,k_n)}(x_1, x_2, \ldots, x_n)$, labeled by the dominant n-tuple $(k_1, k_2, \ldots, k_n)$ with $k_1 \ge k_2 \ge \cdots \ge k_n$ and $k_j \ge 0$, is in the form

$$P^+_{(k_1+n,k_2,\ldots,k_{n-1},0)} + \sum_{j=1}^{n} (-1)^j \frac{X_j}{j!(n-j)!} \, P^+_{(k_1+n-j,k_2,\ldots,k_{n-1},0)} = 0, \tag{15}$$

where variables X_j are connected with the lowest n symmetric exponential functions

$$X_j = E^+_{(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{n-j})}, \qquad j = 1, \ldots, n. \tag{16}$$

The characteristic polynomial equation corresponding to the equation (15) is of the form

$$\lambda^n + \sum_{j=1}^{n} (-1)^j \frac{X_j}{j!(n-j)!} \, \lambda^{n-j} = 0 \tag{17}$$

and general solution of (15) has the following form

$$P^+_{(k_1,k_2,\ldots,k_{n-1},0)} = \sum_{j=1}^{n} c_j^{(k_2,\ldots,k_{n-1})} \lambda_j^{k_1}, \tag{18}$$

where λ_j are roots of the characteristic equation (17). Since polynomials $P^+_{(k_1,k_2,\ldots,k_{n-1},k_n)}$ are symmetric with respect to all permutations of indices, coefficients $c_j^{(k_2,\ldots,k_{n-1})}$ can be found (taking into account the initial conditions).

Finally for $P^+_{(k_1,k_2,\ldots,k_{n-1},0)}$ we get

$$P^+_{(k_1,k_2,\ldots,k_{n-1},0)} = \sum_{\substack{i_1,\ldots,i_{n-1}\in\{1,\ldots,n\} \\ i_l \ne i_t,\, l\ne t}} \lambda_{i_1}^{k_1} \ldots \lambda_{i_{n-1}}^{k_{n-1}}.$$

Since polynomials $P^+_{(k_1,k_2,\ldots,k_{n-1},k_n)}$ fulfil the relation

$$P^+_{(k_1,k_2,\ldots,k_{n-1},k_n)} = \left(\frac{X_n}{n!}\right)^{k_n} P^+_{(k_1-k_n,k_2-k_n,\ldots,k_{n-1}-k_n,0)}$$

the explicit formula for polynomials $P^+_{(k_1,k_2,\ldots,k_{n-1},k_n)}$ reads as

$$P^+_{(k_1,k_2,\ldots,k_{n-1},k_n)} = \left(\frac{X_n}{n!}\right)^{k_n} \sum_{\substack{i_1,\ldots,i_{n-1}\in\{1,\ldots,n\} \\ i_l \ne i_t,\, l\ne t}} \lambda_{i_1}^{k_1-k_n} \ldots \lambda_{i_{n-1}}^{k_{n-1}-k_n}.$$

4 Four dimensional antisymmetric exponential functions

The four dimentional antisymmetric exponential functions $E^-_{(\lambda,\mu,\nu,\xi)}\colon \mathbb{R}^4 \to \mathbb{C}$ are defined in similar way to the symmetric case by the explicit formula

$$\begin{aligned}
E^-_{(\lambda,\mu,\nu,\xi)}(x,y,z,t) &= \begin{vmatrix} e^{2\pi i\lambda x} & e^{2\pi i\lambda y} & e^{2\pi i\lambda z} & e^{2\pi i\lambda t} \\ e^{2\pi i\mu x} & e^{2\pi i\mu y} & e^{2\pi i\mu z} & e^{2\pi i\mu t} \\ e^{2\pi i\nu x} & e^{2\pi i\nu y} & e^{2\pi i\nu z} & e^{2\pi i\nu t} \\ e^{2\pi i\xi x} & e^{2\pi i\xi y} & e^{2\pi i\xi z} & e^{2\pi i\xi t} \end{vmatrix} \\
&= e^{2\pi i(\lambda x+\mu y+\nu z+\xi t)} + e^{2\pi i(\lambda x+\mu z+\nu t+\xi y)} + e^{2\pi i(\lambda x+\mu t+\nu y+\xi z)} \\
&- e^{2\pi i(\lambda x+\mu t+\nu z+\xi y)} - e^{2\pi i(\lambda x+\mu y+\nu t+\xi z)} - e^{2\pi i(\lambda x+\mu z+\nu y+\xi t)} \\
&- e^{2\pi i(\lambda y+\mu x+\nu z+\xi t)} - e^{2\pi i(\lambda y+\mu t+\nu x+\xi z)} - e^{2\pi i(\lambda y+\mu z+\nu t+\xi x)} \\
&+ e^{2\pi i(\lambda y+\mu t+\nu z+\xi x)} + e^{2\pi i(\lambda y+\mu x+\nu t+\xi z)} + e^{2\pi i(\lambda y+\mu z+\nu x+\xi t)} \\
&+ e^{2\pi i(\lambda z+\mu x+\nu y+\xi t)} + e^{2\pi i(\lambda z+\mu y+\nu t+\xi x)} + e^{2\pi i(\lambda z+\mu t+\nu x+\xi y)} \\
&- e^{2\pi i(\lambda z+\mu t+\nu y+\xi x)} - e^{2\pi i(\lambda z+\mu x+\nu t+\xi y)} - e^{2\pi i(\lambda z+\mu y+\nu x+\xi t)} \\
&- e^{2\pi i(\lambda t+\mu x+\nu y+\xi z)} - e^{2\pi i(\lambda t+\mu z+\nu x+\xi y)} - e^{2\pi i(\lambda t+\mu y+\nu z+\xi x)} \\
&+ e^{2\pi i(\lambda t+\mu z+\nu y+\xi x)} + e^{2\pi i(\lambda t+\mu x+\nu z+\xi y)} + e^{2\pi i(\lambda t+\mu y+\nu x+\xi z)}.
\end{aligned} \tag{19}$$

The functions $E^-_{(\lambda,\mu,\nu,\xi)}$ are antisymmetric with respect to odd permutations of the variables (x,y,z,t) and symmetric with respect to even permutations,

$$\begin{aligned}
&E^-_{(\lambda,\mu,\nu,\xi)}(x,y,z,t) = E^-_{(\lambda,\mu,\nu,\xi)}(x,t,y,z) = E^-_{(\lambda,\mu,\nu,\xi)}(x,z,t,y) \\
&= -E^-_{(\lambda,\mu,\nu,\xi)}(x,y,t,z) = -E^-_{(\lambda,\mu,\nu,\xi)}(x,t,z,y) = -E^-_{(\lambda,\mu,\nu,\xi)}(x,z,y,t) \\
&= -E^-_{(\lambda,\mu,\nu,\xi)}(t,x,y,z) = -E^-_{(\lambda,\mu,\nu,\xi)}(t,z,x,y) = -E^-_{(\lambda,\mu,\nu,\xi)}(t,y,z,x) \\
&= E^-_{(\lambda,\mu,\nu,\xi)}(t,x,z,y) = E^-_{(\lambda,\mu,\nu,\xi)}(t,z,y,x) = E^-_{(\lambda,\mu,\nu,\xi)}(t,y,x,z) \\
&= E^-_{(\lambda,\mu,\nu,\xi)}(z,t,x,y) = E^-_{(\lambda,\mu,\nu,\xi)}(z,y,t,x) = E^-_{(\lambda,\mu,\nu,\xi)}(z,x,y,t) \\
&= -E^-_{(\lambda,\mu,\nu,\xi)}(z,t,y,x) = -E^-_{(\lambda,\mu,\nu,\xi)}(z,y,x,t) = -E^-_{(\lambda,\mu,\nu,\xi)}(z,x,t,y) \\
&= -E^-_{(\lambda,\mu,\nu,\xi)}(y,z,t,x) = -E^-_{(\lambda,\mu,\nu,\xi)}(y,x,z,t) = -E^-_{(\lambda,\mu,\nu,\xi)}(y,t,x,z) \\
&= E^-_{(\lambda,\mu,\nu,\xi)}(y,z,x,t) = E^-_{(\lambda,\mu,\nu,\xi)}(y,x,t,z) = E^-_{(\lambda,\mu,\nu,\xi)}(y,t,z,x).
\end{aligned} \tag{20}$$

Similarly, they are symmetric and antisymmetric, with respect to even and odd permutations of the indices (λ,μ,ν,ξ), when the functions $E^-_{(\lambda,\mu,\nu,\xi)}$ are evaluated at a fixed point (x,y,z,t). Therefore only $E^-_{(\lambda,\mu,\nu,\xi)}$ with the so called strictly dominant quadruple (λ,μ,ν,ξ) can be considered, that is with $\lambda > \mu > \nu > \xi$. The functions $E^-_{(k,l,m,n)}$ with $k,l,m,n \in \mathbb{Z}$ satisfy the invariance with respect to integer shifts

$$E^-_{(k,l,m,n)}(x+r, y+s, z+u, t+w) = E^-_{(k,l,m,n)}(x,y,z,t), \quad r,\ s,\ u, w \in \mathbb{Z}. \tag{21}$$

Due to their property of symmetry and antisymmetry, with respect to even and odd permutations and (21), the functions $E^-_{(k,l,m,n)}$ with $k,\ l,\ m, n \in \mathbb{Z}$ can be considered on the fundamental domain $F(S_4^{\mathrm{aff}})$ only.

4.1 Orthogonal polynomials corresponding to antisymmetric exponential functions

The orthogonal polynomials corresponding to antisymmetric exponential functions of four variables are constructed in a similar way as in symmetric case. Four variable polynomials $P^-_{(k,l,m,n)}$ with the elementary variables (6) and strictly dominant quadruples (k, l, m, n) are defined by the condition

$$P^-_{(k,l,m,n)}(X(x,y,z,t),Y(x,y,z,t),Z(x,y,z,t),T(x,y,z,t)) = \frac{E^-_{(k,l,m,n)}(x,y,z,t)}{E^-_{(3,2,1,0)}(x,y,z,t)}.$$

First polynomials $P^-_{(k,l,m,n)}(X,Y,Z,T)$ are given in the following table:

$$\begin{aligned}
P^-_{(3,2,1,0)} &= 1 & P^-_{(5,4,2,0)} &= \frac{1}{24}YZ - \frac{1}{144}T \\
P^-_{(4,2,1,0)} &= \frac{1}{6}X & P^-_{(5,4,2,1)} &= \frac{1}{96}YT \\
P^-_{(4,3,1,0)} &= \frac{1}{4}Y & P^-_{(5,4,3,0)} &= \frac{1}{36}Z^2 - \frac{1}{96}YT
\end{aligned}$$

$$\begin{aligned}
P^-_{(4,3,2,0)} &= \frac{1}{6}Z & P^-_{(5,4,3,1)} &= \frac{1}{144}ZT \\
P^-_{(4,3,2,1)} &= \frac{1}{24}T & P^-_{(5,4,3,2)} &= \frac{1}{24^2}T^2 \\
P^-_{(4,3,2,0)} &= \frac{1}{6}Z & P^-_{(6,2,1,0)} &= \frac{1}{6^3}X^3 - \frac{1}{12}XY + \frac{1}{6}Z \\
P^-_{(5,3,1,0)} &= \frac{1}{24}(XY - 4Z) & P^-_{(6,3,1,0)} &= \frac{1}{144}X^2Y - \frac{1}{16}Y^2 - \frac{1}{36}XZ + \frac{1}{24}T \\
P^-_{(5,3,2,0)} &= \frac{1}{36}XZ - \frac{1}{24}T & P^-_{(6,3,2,0)} &= \frac{1}{6^3}X^2Z - \frac{1}{24}YZ - \frac{1}{144}XT \\
P^-_{(5,3,2,1)} &= \frac{1}{144}XT & P^-_{(6,3,2,1)} &= \frac{1}{864}X^2T - \frac{1}{96}YT \\
P^-_{(5,4,1,0)} &= \frac{1}{16}Y^2 - \frac{1}{36}XZ & P^-_{(6,4,1,0)} &= \frac{1}{144}XY^2 - \frac{1}{36}YZ - \frac{1}{6^3}X^2Z + \frac{1}{144}XT
\end{aligned}$$

Generating function of $P^-_{(k,l,m,n)}$ The antisymmetric generating function is given explicitly as

$$\begin{aligned}
G^-(X,Y,Z,T,r,s,u,w) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} P^-_{(k,l,m,n)}(X,Y,Z,T) r^k s^l u^m w^n \\
&= \frac{331776\,(r-s)(s-u)}{(r^4T-4r^3Z+6r^2Y-4rX+24)(s^4T-4s^3Z+6s^2Y-4sX+24)} \\
&\times \frac{(u-w)(w-r)}{(u^4T-4u^3Z+6u^2Y-4uX+24)(w^4T-4w^3Z+6w^2Y-4wX+24)}.
\end{aligned}$$

Continuous orthogonality of $P^-_{(k,l,m,n)}$. The polynomials $P^-_{(k,l,m,n)}$ and $P^-_{(k',l',m',n')}$ are continuously orthogonal and the orthogonality holds for any two dominant quadruples (k,l,m,n), $(k',l',m',n') \in (\mathbb{Z}^{\geq 0})^4$, with respect to some weight function $\varrho(X,Y,Z,T)/J(X,Y,Z,T)$. The orthogonality relations are then of the form

$$\begin{aligned}
\int_{F_4} \frac{\varrho(X,Y,Z,T)}{J(X,Y,Z,T)} P^-_{(k,l,m,n)}(X,Y,Z,T)\overline{P^-_{(k',l',m',n')}(X,Y,Z,T)}\mathrm{d}X\mathrm{d}Y\mathrm{d}Z\mathrm{d}T \\
= \delta_{kk'}\delta_{ll'}\delta_{mm'}\delta_{nn'},
\end{aligned}$$

where F_4 is the transformed fundamental domain $F(S_4^{\text{aff}})$ via mapping (6).

Discrete orthogonality of $P^-_{(k,l,m,n)}$. Simirarly, the discrete orthogonality of the polynomials $P^-_{(k,l,m,n)}$, $P^-_{(k',l',m',n')}$ with respect to the weight function ϱ holds for any (k,l,m,n), (k',l',m',n') and is of the form

$$\begin{aligned}
\sum_{(X_r,Y_s,Z_u,T_w)\in\mathcal{L}_N^-} \varrho(X_r,Y_s,Z_u,T_w) P^-_{(k,l,m,n)}(X_r,Y_s,Z_u,T_w)\overline{P^-_{(k',l',m',n')}(X_r,Y_s,Z_u,T_w)} \\
= N^4\delta_{kk'}\delta_{ll'}\delta_{mm'}\delta_{nn'}
\end{aligned}$$

where $\mathcal{L}_N^-$ is the discrete set of points, which is the grid L_N^- transformed via the mapping (6), given as

$$\begin{aligned}
\mathcal{L}_N^- = \{(X(x_r,y_s,z_u,t_w),Y(x_r,y_s,z_u,t_w),Z(x_r,y_s,z_u,t_w),T(x_r,y_s,z_u,t_w)) \\
\mid (x_r,y_s,z_u,t_w)\in L_N^-\}.
\end{aligned}$$

The grid L_N^-, with its points contained inside the closure of the fundamental domain $F(S_4^{\text{aff}})$, is given explicitly as

$$L_N^- = \left\{\left(\frac{r}{N},\frac{s}{N},\frac{u}{N},\frac{w}{N}\right) \,\middle|\, r>s>u>w;\ r,s,u,w=0,1,2,...,N-1\right\}. \quad (22)$$

The number N fixes the density of the grid inside $\overline{F(S_4^{\text{aff}})}$.

5 General formula for polynomials $P^-_{(k_1,k_2,\ldots,k_n)}$

The orthogonal polynomials of n variables $P^-_{(k_1,k_2,\ldots,k_{n-1},k_n)}$ corresponding to antisymmetric exponential functions with the elementary variables (16) and strictly dominant n-tuples $(k_1,k_2,\ldots,k_{n-1},k_n)$, $k_1 > k_2.\cdots > k_n$ and $k_j \geq 0$ are defined by the condition

$$P^-_{(k_1,k_2,\ldots,k_{n-1},k_n)} = \frac{E^-_{(k_1,k_2,\ldots,k_{n-1},k_n)}}{E^-_{(n-1,n-2,\ldots,1,0)}}.$$

Applying similar arguments as in Section 3, one can derive the explicit formula for the polynomials $P^-_{(k_1,k_2,\ldots,k_{n-1},k_n)}$,

$$\begin{aligned} P^-_{(k_1,k_2,\ldots,k_{n-1},k_n)} = \left(\frac{X_n}{n!}\right)^{k_n} & H(X_1,X_2,\ldots,X_n)\times \\ \times & \sum_{\substack{i_1,\ldots,i_{n-1}\in\{1,\ldots,n\} \\ i_l\neq i_t,\, l\neq t}} c_{i_1,\ldots,i_{n-1}}\, \lambda_{i_1}^{k_1-k_n}\ldots\lambda_{i_{n-1}}^{k_{n-1}-k_n}, \end{aligned}$$

where H is a function depended on roots of the characteristic polynomial equation (17), and the coefficient $c_{i_1,\ldots,i_{n-1}}$ depends on permutations of roots $\lambda_1,\ldots,\lambda_n$.

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References

1. A. Bezubik, J. Hrivnák, S. Pošta *Two dimensional symmetric and antisymmetric exponential functions and orthogonal polynomials*, Journal of Physics: Conference Series, Volume 411, Paper 012007 (2013),
2. A. Bezubik, J. Hrivnák, J. Patera, S. Pošta *Three-variable symmetric and antisymmetric exponential functions and orthogonal polynomials* (in press)
3. G. J. Heckman, E. M. Opdam, *Root systems and hypergeometric functions. I,* Compositio Mathematica, **64** (1987), 329–352
4. J. Hrivnák, J. Patera, *Two-dimensional symmetric and antisymmetric generalizations of exponential and cosine functions,* J. Math. Phys. **51** (2010), 023515; arXiv:0911.4209
5. J. Hrivnák, J. Patera, S. Pošta, *Three-variable exponential functions of the alternating group,* J. Phys. A: Math. Theor. **45** (2012) 045201–045210
6. A. Klimyk, J. Patera: *(Anti)symmetric multivariate exponential functions and corresponding Fourier transforms*, J. Phys. A: Math. Theor. **40** (2007), 10473–10489

7. A.U. Klimyk, J. Patera, *Alternating group and multivariate exponential functions,* in *Groups and Symmetries; From Neolithic Scots to John McKay,* AMS-CRM Proceedings and Lectures Notes Series, **47**, 233-246, eds. J. Harnad and P. Winternitz; arXiv:0907.0601v1
8. H. Minc, *Permanents,* Addison-Wesley, New York 1978

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Relative Observability, Duality for Fractional Differential-algebraic Delay Systems with Jumps

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Abstract. The problem of relative R-observability is considered for linear stationary fractional differential-algebraic delay system with jumps (FDADJ). FDADJ system consists of fractional differential equation in the Caputo Sense and an output equation. We introduce the determining equation systems and their properties. We derive solutions representations into series of their determining equation solutions and obtain effective parametric rank criteria for relative R-observability. A dual controllability result is also formulated.

Keywords: Fractional differential equations, determining equations, differential-algebraic systems

1 Introduction

Observability is one of the fundamental concepts in modern mathematical control theory. This concept was introduced by R. Kalman in 1960. This is qualitative property of observation systems and is of great importance in control theory. The basic concepts of observability play an important role in dynamical systems analysis.

During last few years many results concerning observability of fractional systems both discrete-time and continuous-time have been published in the literature (see e.g. [1]–[2]). It should be pointed out, that for linear systems most observability results are dual with corresponding controllability results.

The paper deals with linear fractional differential-algebraic delay systems with jumps (FDADJ). FDADJ systems consist of some equations being fractional differential in the Caputo sense, the other – difference and discrete. For the fractional differential equation we introduce impacts (impulses) defined by discrete equation. We introduce the determining equations the same as for differential-algebraic systems (for example see [8] or [9]). To obtain solutions representations we apply fractional differential calculus especially dealing with the Laplace transform. By this result we obtain effective parametric rank criterions for $\mathbb{R}$-observability with respect to the discrete variable. The relative $\mathbb{R}$-observability is dual to relatively controllability with respect to x_2 (see [12] for more details).

Our results can be considered as a generalization of the known corresponding results for the integer order case to the fractional order one, since for $\alpha = 1$, the results for the integer case are recovered (see [6] for the nonstationary case). Observe that for differential-algebraic systems with delay (DAD) some kinds of neutral type time-delay and discrete-continuous hybrid systems can be regarded as examples of DAD systems [7].

The paper is organized as follows. In section 2 the state equations of FDADJ systems, their deterministic equations and representation of solutions into series of determining equations solutions are introduced. The observability problem is analyzed in section 3. The duality result is investigated in section 4. Finally, section 5 contains simple numerical example, which illustrates theoretical considerations.

2 Observation System

Let us introduce the following notation:

${}^{C}D_t^{\alpha}$ is the left-sided Caputo fractional derivatives of order α defined by

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \left(\frac{d}{dt} f(\tau) \right) d\tau, $$

where $0 < \alpha < 1$, $\alpha \in \mathbb{R}$ and $\Gamma(t) = \int_0^\infty e^{-\tau} \tau^{t-1} d\tau$ is the Euler gamma function (see [4] for more details). $T_t = \lim_{\varepsilon \to +0} \left[\frac{t-\varepsilon}{h} \right]$, where the symbol $[z]$ means entire part of the number z, h is defined below; I_n is the identity n by n matrix.

In this paper, we introduce the stationary FDADJ observation system in the following form:

$$ {}^{C}D_t^{\alpha} x_1(t) = A_{11} x_1(t) + A_{12} x_2(t),\ t > 0, \tag{1} $$

$$ x_2(t) = A_{21} x_1(t) + A_{22} x_2(t-h),\ t \geq 0, \tag{2} $$

$$ x_1(jh) - x_1(jh-0) = A_{12} x_3(jh-h),\ j = 1...T_t, \tag{3} $$

$$ x_3(t) = A_{22} x_3(t-h),\ t \geq h, \tag{4} $$

$$ y_1(t) = B_1 x_1(t) + B_2 x_2(t), \tag{5} $$

$$ y_2(t) = B_2 x_3(t), \tag{6} $$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $x_3(t) \in \mathbb{R}^{n_2}$, $y_1(t) \in \mathbb{R}^r$, $y_2(t) \in \mathbb{R}^r$, $t \geq 0$, $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{r \times n_1}$, $B_2 \in \mathbb{R}^{r \times n_2}$ are constant (real) matrices, $0 < h$ is a constant delay. We regard an absolute continuous n_1-vector function $x_1(\cdot)$ on the intervals $[jh, (j+1)h)$, $j = 1...T_t$ and a piecewise continuous n_2-vector function $x_2(\cdot)$ and discrete value function $x_3(\cdot)$ as a solution of system (1)-(6) for almost all $t > 0$. Equation 3 is defined only for discrete points $t = jh$, $j = 1...T_t$, where $T_t h \leq t$ and defines values of jumps of function $x_1(\cdot)$ in these points.

System (1)-(6) should be completed with initial conditions:

$$ x_1(0+) = x_1(0) = x_{10},\ x_3(0) = x_{30}, x_3(\xi) = 0, \text{ for } \xi < 0, $$
$$ x_2(\tau) = \psi(\tau),\ \tau \in [-h, 0), \tag{7} $$

where $\psi \in PC([-h,0],\mathbb{R}^{n_2})$. Let us introduce the determining equations of system (1)-(6) (see [9] for more details).

$$X_{1,k}(t) = A_{11}X_{1,k-1}(t) + A_{12}X_{2,k-1}(t) + U_{k-1}(t), \tag{8}$$
$$X_{2,k}(t) = A_{21}X_{1,k}(t) + A_{22}X_{2,k}(t-h), \tag{9}$$
$$X_3(t) = A_{22}X_3(t-h), \tag{10}$$
$$Y_k(t) = B_1X_{1,k}(t) + B_2X_{2,k}(t), \tag{11}$$

for $k = 0, 1, ...;\ t \geq 0$ with initial conditions

$$X_{1,k}(t) = 0, X_{2,k}(t) = X_3(t) = 0,\ Y_k(t) = 0 \text{ for } t < 0 \text{ or } k \leq 0;\ X_3(0) = I_{n_2},$$
$$U_0(0) = I_n, U_k(t) = 0 \text{ for } t^2 + k^2 \neq 0.$$

It is easy to see that $X_{1,k}(t) = 0,\ X_{2,k}(t) = 0,\ X_3(t) = 0,\ Y_k(t) = 0$ for $t \neq jh$, where $j = 0, 1, \ldots$ i $k = 0, 1, \ldots$ Here, we establish some algebraic properties of $X_{1,k},\ X_{2,k}$.

Proposition 1 *The following identities hold [12]*

$$\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k = \sum_{j=0}^{+\infty} X_{1,k+1}(jh)\omega^j,\ k = 1, 2, \ldots;$$

$$(I_{n_2} - \omega A_{22})^{-1}A_{12}\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k$$
$$= \sum_{j=0}^{+\infty} X_{2,k+1}(jh)\omega^j,\ k = 1, 2, \ldots;$$

where $|\omega| < \omega_1$ and ω_1 is a sufficiently small real number.

Now we can formulate.

Theorem 2 *A solution to system (1)-(6) with initial conditions (7) for $t \geq 0$ exists, is unique and can be represented in the form of a series in power of solutions to determining system (8)-(11) in the following form*

$$x_1(t)$$
$$= \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j)h>0}} X_{1,k+1}(jh)A_{12}(A_{22})^{i+1} \int_0^{t-(i+j)h} \frac{(t-\tau-(i+j)h)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}\psi(\tau-h)d\tau$$
$$+ \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} X_{1,k+1}(jh)A_{12}X_3(ih)x_{30} \tag{12}$$
$$+ \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha k}}{\Gamma(\alpha k+1)} X_{1,k+1}(jh)x_{10},$$

$$
\begin{aligned}
&x_2(t)\\
&=\sum_{k=0}^{+\infty}\sum_{\substack{i,j\\ t-(i+j)h>0}} X_{2,k+1}(jh)A_{12}(A_{22})^{i+1}\int_0^{t-(i+j)h}\frac{(t-\tau-(i+j)h)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}\psi(\tau-h)d\tau\\
&+\sum_{k=0}^{+\infty}\sum_{\substack{i,j\\ t-(i+j+1)h>0}}\frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)}X_{2,k+1}(jh)A_{12}X_3(ih)x_{30} \qquad (13)\\
&+\sum_{i=0}^{+\infty}(A_{22})^{i+1}\psi(t-(i+1)h)+\sum_{k=0}^{+\infty}\sum_{\substack{j\\ t-jh>0}}\frac{(t-jh)^{\alpha k}}{\Gamma(\alpha k+1)}X_{2,k+1}(jh)x_{10},
\end{aligned}
$$

where $\psi(\tau)=0$ *for* $\tau\notin[-h,0]$.

Proof. First we use the classical formula for the Laplace transformation of ${}^CD_t^\alpha x_1(\cdot)$

$$
\begin{aligned}
&\int_0^\infty e^{-st}\ {}^CD_t^\alpha x_1(t)dt=s^{-(1-\alpha)}\int_0^{+\infty}\dot{x}_1(\tau)e^{-s\tau}d\tau\\
&=s^{-(1-\alpha)}\sum_{k=0}^{+\infty}\int_{kh}^{kh+h}\dot{x}_1(\tau)e^{-s\tau}d\tau=s^{-(1-\alpha)}\sum_{k=0}^{+\infty}\Big(x_1(kh+h-0)e^{-s(kh+h)}\\
&-x_1(kh+0)e^{-s(kh)}+s\int_{kh}^{kh+h}x_1(\tau)e^{-s\tau}d\tau\Big)=s^{\alpha-1}\lim_{k\to\infty}x_1(kh)e^{-s(kh)}\\
&-s^{\alpha-1}x_{10}+s^{\alpha-1}\sum_{k=1}^{+\infty}\Big[x_1(kh-0)e^{-s(kh)}-x_1(kh+0)e^{-s(kh)}\Big]\\
&+s^\alpha\int_0^{+\infty}x_1(\tau)e^{-s\tau}d\tau=s^\alpha\breve{x}(s)-s^{\alpha-1}x_{10}-s^{\alpha-1}\sum_{k=1}^{+\infty}A_{12}x_3(kh-h)e^{-s(kh)}\\
&=s^\alpha\breve{x}_1(s)-s^{\alpha-1}x_{10}-s^{\alpha-1}A_{12}e^{-sh}\sum_{k=0}^{+\infty}(A_{22})^k x_{30}e^{-s(kh)}\\
&=s^\alpha\breve{x}_1(s)-s^{\alpha-1}x_{10}-s^{\alpha-1}A_{12}e^{-sh}(\sum_{k=0}^{+\infty}X_3(kh)x_{30}e^{-s(kh)}).
\end{aligned}
$$

We apply the Laplace transform to (2), (1) and substitute the above

$$
\breve{x}_2(s)=(I_{n_2}-A_{22}\omega)^{-1}A_{21}\breve{x}_1(s)+(I_{n_2}-A_{22}\omega)^{-1}A_{22}e^{-sh}\int_{-h}^0 e^{-s\tau}\psi(\tau)d\tau,
$$

$$\begin{aligned}
\breve{x}_1(s) &= \left[s^\alpha I_{n_1} - A_{11} - A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{21}\right]^{-1}\left(A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{22}e^{-sh}\right.\\
&\quad \left.\times \int_{-h}^{0} e^{-s\tau}\psi(\tau)d\tau + s^{\alpha-1}x_{10} + s^{\alpha-1}A_{12}e^{-sh}\sum_{i=0}^{+\infty} X_3(ih)x_{30}e^{-s(ih)}\right)\\
&= \sum_{k=0}^{+\infty}\frac{1}{(s^\alpha)^{k+1}}\left[A_{11} + A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{21}\right]^k\left(A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{22}e^{-sh}\right.\\
&\quad \left.\times \int_{-h}^{0} e^{-s\tau}\psi(\tau)d\tau + s^{\alpha-1}x_{10} + s^{\alpha-1}A_{12}e^{-sh}\sum_{i=0}^{+\infty} X_3(ih)e^{-s(ih)}x_{30}\right).
\end{aligned}$$

Applying Propositions 1 we obtain

$$\begin{aligned}
\breve{x}_1(s) &= \sum_{k=0}^{+\infty}\frac{1}{s^{\alpha k+\alpha}}\sum_{j=0}^{+\infty} e^{-jsh}X_{1,k+1}(jh)A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{22}e^{-sh}\\
&\times \int_{-h}^{0} e^{-s\tau}\psi(\tau)d\tau + \sum_{k=0}^{+\infty}\frac{1}{s^{\alpha k+1}}\sum_{j=0}^{+\infty} e^{-jsh}X_{1,k+1}(jh)x_{10}\\
&+ \sum_{k=0}^{+\infty}\frac{1}{s^{\alpha k+1}}\sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty} e^{-(i+j+1)sh}X_{1,k+1}(jh)A_{12}X_3(ih)x_{30},\\
\breve{x}_2(s) &= \sum_{k=0}^{+\infty}\frac{1}{s^{\alpha k+\alpha}}\sum_{j=0}^{+\infty} e^{-jsh}X_{2,k+1}(jh)A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{22}e^{-sh}\\
&\times \int_{-h}^{0} e^{-s\tau}\psi(\tau)d\tau + \sum_{k=0}^{+\infty}\frac{1}{s^{\alpha k+1}}\sum_{j=0}^{+\infty} e^{-jsh}X_{2,k+1}(jh)x_{10}\\
&+ \sum_{k=0}^{+\infty}\frac{1}{s^{\alpha k+1}}\sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty} e^{-(i+j+1)sh}X_{2,k+1}(jh)A_{12}X_3(ih)x_{30}\\
&\qquad + (I_{n_2} - A_{22}\omega)^{-1}A_{22}e^{-sh}\int_{-h}^{0} e^{-s\tau}\psi(\tau)d\tau.
\end{aligned}$$

By applying inverse Laplace transform the proof is complete. □

3 Observability

Now we present algebraic properties of solutions of determining equations: $Y_k(t)$ i $X_3(t)$.

Proposition 3 *The following identity holds [13]:*

$$\sum_{i=0}^{\xi} Y_\eta((\xi-i)h)A_{12}X_3(ih) = -\sum_{j=1}^{\theta_\xi} r_{0j} \sum_{i=0}^{\xi-j} Y_\eta((\xi-j-i)h)A_{12}X_3(ih) \tag{14}$$
$$-\sum_{k=1}^{n_1}\sum_{j=0}^{\theta_\xi} r_{ij} \sum_{i=0}^{\xi-j} Y_{\eta-k}((\xi-j-i)h)A_{12}X_3(ih),$$

for $\eta = n_1+1, n_1+2, ...$ *and* $\xi = 0,1,...;$ $\theta_\xi = \min\{\xi,\, n_1n_2\}$.

Similar to the above:

Proposition 4 *Solutions* $Y_k(\cdot)X_3(\cdot)$, *of determining equations (10), (11) satisfy the following conditions [13]:*

$$\sum_{i=0}^{\nu} Y_k((\nu-i)h)A_{12}X_3(ih) \tag{15}$$
$$= -\sum_{j=1}^{\min\{k-1,n_1n_2^2\}} r_{0j}\sum_{i=0}^{\nu} Y_{k-j}((\nu-i)h)A_{12}X_3(ih)$$
$$-\sum_{l=1}^{n_2}\sum_{j=0}^{\min\{k-1,n_1n_2^2\}} r_{ij}\sum_{i=0}^{\nu-l} Y_{k-j}((\nu-l-i)h)A_{12}X_3(ih)$$

for $\nu = n_2+1, n_2+2, ...$ *and* $k = 1, 2, ...$

Let $x_1(t,x_{10},\psi,x_{30})$, $x_2(t,x_{10},\psi,x_{30}), x_3(t,x_{10},\psi,x_{30})$ be the solutions of system (1)-(6) for $t \geq 0$ corresponding to initial data (7). Similarly, $y_1(t) = y_1(t,x_{10},\psi,x_{30})$, $y_2(t) = y_2(t,x_{10},\psi,x_{30})$, $\tilde{y}_1(t) = \tilde{y}_1(t,x_{10},\psi,\tilde{x}_{30})$, $\tilde{y}_2(t) = \tilde{y}_2(t,x_{10},\psi,\tilde{x}_{30})$, denote the outputs corresponding to the solutions $x_1(t,x_{10},\psi,x_{30})$, $x_2(t,x_{10},\psi,x_{30})$, $x_3(t,x_{10},\psi,x_{30})$ and $\tilde{x}_1(t,x_{10},\psi,\tilde{x}_{30})$, $\tilde{x}_2(t,x_{10},\psi,\tilde{x}_{30})$, $\tilde{x}_3(t,x_{10},\psi,\tilde{x}_{30})$, respectively.

Definition 5 *System (1)-(6) is said to be* $\mathbb{R}^n$*-observable with respect to* x_3 *if for every* x_{30}, $\tilde{x}_{30} \in \mathbb{R}^{n_2}$ *the condition*

$$y_1(t,x_{10},\psi,x_{30}) = \tilde{y}_1(t,x_{10},\psi,\tilde{x}_{30}),\;\; y_2(t,x_{10},\psi,x_{30}) = \tilde{y}_2(t,x_{10},\psi,\tilde{x}_{30})$$

for every $\psi \in PC([-h,0),\mathbb{R}^{n_2})$, $x_{10} \in \mathbb{R}^{n_1}$ and for $t > 0$ *implies that* $x_{30} = \tilde{x}_{30}$.

Lemma 1 *Let define functions* $f_{ijk}(t)$ *as* $f_{ijk}(t) = \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)}$ *for* $t-(i+j)h > 0$ *and* $f_{ijk}(t) = 0$ *for* $t-(i+j)h < 0$, *then* $\sum_{i+j=l} f_{ijk}(t)$ for $l = 0,1,\ldots;$ $k = 0,1,\ldots$ *are linearly independent for* $t > 0$.

Proof. For $t > 0$, $t \in [(i+j+1)h,\ (i+j+2)h)$ we define $\alpha_{ijk} \in \mathbb{R}$ and $l = i+j$ then $t \in [(l+1)h,\ (l+2)h)$. For $t \geq 0$, $t \in [(l+1)h, (l+2)h)$, $\ l = 0$, assume that

$$\sum_{\substack{k=0\\ i+j=0}}^{+\infty} \alpha_{ijk} \frac{(t-h)^{\alpha k}}{\Gamma(\alpha k+1)} \equiv 0,\ t \in [h, 2h),\ \alpha_{ijk} \in \mathbb{R}.$$

Let $t \to h^+$, then $\sum_{i+j=0} \alpha_{ij0} = 0$. This implies

$$\sum_{\substack{k=1\\ i+j=0}}^{+\infty} \alpha_{ijk} \frac{(t-h)^{\alpha k}}{\Gamma(\alpha k+1)} \equiv 0,\ t \in [h, 2h),$$

and $\sum_{i+j=0} \alpha_{ij1} = 0$. Analogously $\sum_{i+j=0} \alpha_{ijk} = 0,\ k = 0, 1, \ldots,$ Hence Lemma 6 holds true for $k = 0, 1, \ldots$. Then, the proof is by induction on l. □

Theorem 6 *System (1)-(6) is $\mathbb{R}^n$-observable with respect to x_3 if and only if*

$$rank \begin{bmatrix} B_2 X_3(kh) \\ k = 0, \ldots, n_2 \\ \sum_{i=0}^{\xi} Y_\eta((\xi - i)h) A_{12} X_3(ih) \\ \xi = 0, \ldots, n_2;\ \eta = 1, \ldots, n_1; \end{bmatrix} = n_2. \tag{16}$$

Proof. For $y_2(t, x_{10}, \psi, x_{30}) = \tilde{y}_2(t, x_{10}, \psi, \tilde{x}_{30})$ and $x_3(t) = 0, t \neq jh, j = 0, 1, \ldots$ we have

$$B_2 x_3(jh) = B_2 \tilde{x}_3(jh) \Rightarrow B_2(x_3(jh) - \tilde{x}_3(jh)) = 0 \Rightarrow B_2(A_{22})^j(x_{30} - \tilde{x}_{30}) = 0, \quad j = 0, 1, \ldots \tag{17}$$

By the series representation the solutions $x_1(\cdot),\ x_2(\cdot),\ y_1(t, x_{10}, \psi, x_{30})$ $= \tilde{y}_1(t, x_{10}, \psi, \tilde{x}_{30})$ for all $\psi \in PC([-h, 0], \mathbb{R}^{n_2})$, $x_{10} \in \mathbb{R}^{n_1}$ *and for* $t \geq 0$ is equivalent to the following:

$$\begin{aligned}
&B_1 \sum_{k=0}^{+\infty} \sum_{\substack{i,j\\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} X_{1,k+1}(jh) A_{12} X_3(ih) x_{30} \\
&+B_2 \sum_{k=0}^{+\infty} \sum_{\substack{i,j\\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} X_{2,k+1}(jh) A_{12} X_3(ih) x_{30} \\
&= B_1 \sum_{k=0}^{+\infty} \sum_{\substack{i,j\\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} X_{1,k+1}(jh) A_{12} X_3(ih) \tilde{x}_{30} \\
&+ B_2 \sum_{k=0}^{+\infty} \sum_{\substack{i,j\\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} X_{2,k+1}(jh) A_{12} X_3(ih) \tilde{x}_{30}
\end{aligned}$$

It follows from here that

$$\sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} (B_1 X_{1,k+1}(jh) A_{12} X_3(ih)$$
$$+B_2 X_{2,k+1}(jh) A_{12} X_3(ih))\, x_{30}$$
$$= \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} (B_1 X_{1,k+1}(jh) A_{12} X_3(ih)$$
$$+B_2 X_{2,k+1}(jh) A_{12} X_3(ih))\, \tilde{x}_{30}.$$

Then

$$\sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} [\, B_1 \ B_2\,] \begin{bmatrix} X_{1,k+1}(jh) \\ X_{2,k+1}(jh) \end{bmatrix} A_{12} X_3(ih)(x_{30}-\tilde{x}_{30})$$
$$= \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j+1)h>0}} \frac{(t-(i+j+1)h)^{\alpha k}}{\Gamma(\alpha k+1)} Y_{k+1}(jh) A_{12} X_3(ih)(x_{30}-\tilde{x}_{30}) = 0.$$

By Lemma 6 $\sum_{i=0}^{l} Y_{k+1}((l-i)h) A_{12} X_3(ih)$ are linearly independent for $l = 0, 1, \dots;$ $k = 0, 1, \dots$. Moreover by Proposition 3 and Proposition 4 it is easy seen that $\sum_{i=0}^{j} Y_k((j-i)h) A_{12} X_3(ih)$, where $k > n_1 - 1$, $j > n_2$, are a linear combination of $\sum_{i=0}^{\xi} Y_\eta((\xi-i)h) A_{12} X_3(ih)$,$\eta = 1, 2, \dots, n_1;$ $\xi = 0, 1, \dots, n_2$, which completes the proof. □

4 Duality

Let us consider a dual control system

$$\begin{aligned} {}^C D_t^\alpha x_1^*(t) =& A_{11}^T x_1(t) + A_{21}^T x_2^*(t) + B_1^T u(t),\ t>0, \\ x_2^*(t) =& A_{12}^T x_1^*(t) + A_{22} x_2^*(t-h) + B_1^T u(t),\ t \geq 0, \end{aligned} \tag{18}$$

with initial conditions

$$x_1^*(+0) = x_0^*,\ [{}^C D_t^{\alpha-1} x_1^*(t)]_{t=0} = x_0^*,\ x_2^*(\tau) = \psi^*(\tau),\ \tau \in [-h, 0),$$

where $x_1(t) \in \mathbb{R}^{n_1},\ x_2(t) \in \mathbb{R}^{n_2},\ u(t) \in \mathbb{R}^r,\ x_0 \in \mathbb{R}^{n_1};\ \psi \in PC([-h,0), \mathbb{R}^{n_2})$.

Let us consider determining equations

$$\begin{aligned} X_{1,k}^*(t) &= A_{11}^T X_{1,k-1}(t) + A_{21}^T X_{2,k-1}^*(t) + B_1^T U_{k-1}(t), \\ X_{2,k}^*(t) &= A_{12}^T X_{1,k}^*(t) + A_{22}^T X_{2,k}^*(t-h) + B_2^T U_{k-1}(t),\ k = 0, 1, \dots; \end{aligned} \tag{19}$$

with initial conditions

$$X^*_{1,k}(t) = 0, X^*_{2,k}(t) = 0 \text{ for } t < 0 \;\text{ or }\; k \leq 0;$$
$$U^*_0(0) = I_n,\; U^*_k(t) = 0 \;\text{ for }\; t^2 + k^2 \neq 0.$$

Definition 7 *[11]*

Control System (18) is called relatively controllable with respect to x_2 if for any initial data x^*_{10}, φ^* and any $x^*_* \in \mathbb{R}^{n_2}$ there exist a time moment $t_* > 0$ and a piecewise continuous control $u(\cdot)$, such that for the corresponding solution $x^*_2(t) = x^*_2(t, x^*_{10}, \varphi^*, u), t > 0$ the condition $x^*_2(t_*) = x^*_*$ is valid.

The following two statements hold [11].

Proposition 8 *The solutions $X^*_{1,k}(t), X^*_{2,k}(t),\; t \geq 0$ of the determining equations (19) satisfy the following equations:*

$$\sum_{j=0}^{+\infty} X^*_{2,0}(jh)\omega^j \equiv (I_{n_2} - A^T_{22}\omega)^{-1} B^T_2, \tag{20}$$

$$\sum_{j=0}^{+\infty} X^*_{2,k+1}(jh)\omega^j$$
$$\equiv (I_{n_2} - A^T_{22}\omega)^{-1} A^T_{12}(A^T_{11} + A^T_{21}(I_{n_2} - A^T_{22}\omega)^{-1} A^T_{12})^i (B^T_1 + A^T_{21}(I_{n_2} - A^T_{22}\omega)^{-1} B^T_2),$$

where $|\omega| < \omega_1$; ω_1 is a sufficiently small real number.

Proposition 9 *System (18) is relatively controllable with respect to x_2 if and only if*

$$\operatorname{rank}\left[X^*_{2,\eta}(\xi h),\, \xi = 0, \ldots, n_2;\, \eta = 0, \ldots, n_1\right] = n_2, \tag{21}$$

where by the symbol $\left[X^*_{2,\eta}(\xi h),\, \xi = 0, \ldots, n_2;\, \eta = 0, \ldots, n_1\right]$ we denote a block matrix of columns $X^*_{2,\eta}(\xi h)$, for $\xi = 0, \ldots, n_2;\; \eta = 0, \ldots, n_1$.

Now, we can state the duality result.

Theorem 10 *System (1)-(7) is $\mathbb{R}^n$-observable with respect to x_3 if and only if system (18) is relatively controllable with respect to x^*_2.*

Proof. Transposing (20), we have:

$$(B_1 + B_2(I_{n_2} - A_{22}\omega)^{-1} A_{21})(A_{22} + A_{12}(I_m - A_{22}\omega)^{-1} A_{21})^k$$
$$\times (A_{12}(I_{n_2} - A_{22}\omega)^{-1}) = \sum_{j=0}^{+\infty} Y^{*T}_{k+1}(jh)\omega^j,\; k = 0, 1, \ldots; \tag{22}$$
$$B_2(I_{n_2} - A_{22}\omega)^{-1} \equiv \sum_{j=0}^{+\infty} Y^{*T}_0(jh)\omega^j.$$

By (11), Proposition 1 and (10) we obtain:

$$\sum_{l=0}^{+\infty}\sum_{k=0}^{l} Y_{i+1}((l-k)h)A_{12}X_3(kh)\omega^l \equiv$$
$$(B_1+B_2(I_{n_2}-A_{22}\omega)^{-1}A_{21})(A_{11}+A_{12}(I_{n_2}-A_{22}\omega)^{-1}A_{21})^i A_{12}(I_{n_2}-A_{22}\omega)^{-1},$$
$$B_2(I_m - A_{22}\omega)^{-1} = B_2\sum_{\mathrm{k}=0}^{+\infty} X_3(kh)\omega^k. \tag{23}$$

Then, comparing coefficients of the same power of ω in (22) and (23) we have:

$$\begin{aligned} X_{2,0}^{*T}(jh) &= B_2X_3(jh), \\ X_{2,k+1}^{*T}(jh) &= \textstyle\sum_{i=0}^{j} Y_{k+1}((j-i)h)A_{12}X_3(ih). \end{aligned}$$

It follows that

$$\begin{bmatrix} B_2X_3(kh) \\ k=0,\dots,n_2 \\ \sum_{i=0}^{\xi} Y_\eta((\xi-i)h)A_{12}X_3(ih) \\ \xi=0,\dots,n_2;\ \eta=1,\dots,n_1; \end{bmatrix} = \left[X_{2,\eta}^{*}(\xi h), \xi=0,\dots,n_2; \eta=0,\dots,n_1\right]^T.$$

This proves the theorem. □

5 Example

Let us consider the following control system:

$$\begin{aligned} D_t^\alpha x_1(t) =& [1]x_1(t) + \begin{bmatrix}0 & -1\end{bmatrix} x_2(t),\ t>0, \\ x_2(t) =& \begin{bmatrix}0\\1\end{bmatrix} x_1(t) + \begin{bmatrix}0&1\\0&0\end{bmatrix} x_2(t-h),\ t\geq 0, \\ x_1(jh)-x_1(jh-0) =& \begin{bmatrix}0 & -1\end{bmatrix} x_3(jh-h),\ j=1\dots,T_t, \\ x_3(t) =& \begin{bmatrix}0&1\\0&0\end{bmatrix} x_3(t-h), t\geq h, \\ y_1(t) =& [1]x_1(t) + \begin{bmatrix}1 & 0\end{bmatrix} x_2(t), \\ y_2(t) =& \begin{bmatrix}1 & 0\end{bmatrix} x_3(t), \end{aligned} \tag{24}$$

System (24) should be completed with initial conditions:

$$x_{10} = x_{10},\ x_{30} = \begin{bmatrix}x_{31}\\x_{32}\end{bmatrix},\ \psi(\tau) = \begin{bmatrix}1\\1\end{bmatrix},\ \tau\in[-h,0), \tag{25}$$

First we present the determining equations of System (24):

$$X_{1,k}(t) = [1]X_{1,k-1}(t) + \begin{bmatrix}0 & -1\end{bmatrix} X_{2,k-1}(t) + U_{k-1}(t),$$
$$X_{2,k}(t) = \begin{bmatrix}0\\1\end{bmatrix} X_{1,k}(t) + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} X_{2,k}(t-h),$$
$$X_3(t) = A_{22}X_3(t-h),$$
$$Y_k(t) = [1]X_{1,k}(t) + \begin{bmatrix}1 & 0\end{bmatrix} X_{2,k}(t),$$

for $k = 0, 1, ...;\ t \geq 0$ with initial conditions

$$X_{1,k}(t) = 0, X_{2,k}(t) = X_3(t) = 0,\ Y_k(t) = 0 \text{ for } t < 0 \text{ or } k \leq 0;\ X_3(0) = I_{n_2},$$
$$U_0(0) = I_n, U_k(t) = 0 \text{ for } t^2 + k^2 \neq 0.$$

Next compute the solutions of the determining equations of System (24):

$$X_{1,1}(0) = [1],\ X_{2,1}(0) = \begin{bmatrix}0\\1\end{bmatrix},\ X_{1,k}(0) = [0], k \geq 2,\ X_{2,k}(0) = \begin{bmatrix}0\\0\end{bmatrix}, k\ \geq 2,$$
$$X_{1,k}(jh) = [0], k \geq 1, j \geq 1\ X_{2,1}(h) = \begin{bmatrix}1\\0\end{bmatrix},\ X_{2,k}(h) = \begin{bmatrix}0\\0\end{bmatrix}, k\ \geq 2,$$
$$X_{2,k}(jh) = \begin{bmatrix}1\\0\end{bmatrix}, k \geq 1, j \geq 2,$$
$$X_3(0) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ X_3(h) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\ X_3(kh) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, k \geq 2,$$
$$Y_1(0) = [1],\ Y_1(h) = [1],\ Y_1(kh) = [0],\ k \geq 2,$$
$$= \begin{bmatrix}0\\1\\0\end{bmatrix};\ \begin{bmatrix}X_{1,1}(0)\\X_{2,1}(0)\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix};\ \begin{bmatrix}X_{1,k}(0)\\X_{2,k}(0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},\ k \geq 2;$$

Now we preset (12) and (13) for System (24):

Let $0 \leq t < h$

$$x_1(t) = \frac{t^{\alpha\cdot 0}}{\Gamma(1)} X_{1,1}(0)x_{10} = x_{10},$$
$$x_2(t) = \frac{t^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(0)x_{10} + A_{22}\psi(t-h) = \begin{bmatrix}0\\x_{10}\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\x_{10}\end{bmatrix},$$
$$y_1(t) = x_{10} + 1,\ y_2(t) = x_{31},$$

Let $h \leq t < 2h$

$$x_1(t) = \frac{t^{\alpha\cdot 0}}{\Gamma(1)} X_{1,1}(0)x_{10} + \frac{(t-h)^{\alpha\cdot 0}}{\Gamma(1)} X_{1,1}(0)A_{12}X_3(0)x_{30} = x_{10} - x_{32},$$
$$x_2(t) = \frac{t^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(0)x_{10} + \frac{(t-h)^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(h)x_{10} + \frac{(t-h)^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(0)A_{12}X_3(0)x_{30}$$
$$= \begin{bmatrix}0\\x_{10}\end{bmatrix} + \begin{bmatrix}x_{10}\\0\end{bmatrix} + \begin{bmatrix}0\\-x_{32}\end{bmatrix} = \begin{bmatrix}x_{10}\\x_{10} - x_{32}\end{bmatrix},$$

$x_3(t) = \begin{bmatrix} x_{32} \\ 0 \end{bmatrix},\ y_1(t) = 2x_{10} - x_{32},\ y_2(t) = x_{32},$

Let $2h \leq t$

$$x_1(t) = \frac{t^{\alpha\cdot 0}}{\Gamma(1)} X_{1,1}(0)x_{10} + \frac{(t-h)^{\alpha\cdot 0}}{\Gamma(1)} X_{1,1}(0)A_{12}X_3(0)x_{30} = x_{10} - x_{32},$$

$$x_2(t) = \frac{t^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(0)x_{10} + \frac{(t-h)^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(h)x_{10} + \frac{(t-h)^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(0)A_{12}X_3(0)x_{30}$$

$$+ \frac{(t-2h)^{\alpha\cdot 0}}{\Gamma(1)} X_{2,1}(h)A_{12}X_3(0)x_{30} = \begin{bmatrix} 0 \\ x_{10} \end{bmatrix} + \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{32} \end{bmatrix} + \begin{bmatrix} -x_{32} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{10} - x_{32} \\ x_{10} - x_{32} \end{bmatrix},$$

$x_3(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ y_1(t) = 2x_{10} - 2x_{32},\ y_2(t) = 0,$

Let us compute (16):

$$rank \begin{bmatrix} B_2X_3(0) \\ B_2X_3(h) \\ B_2X_3(2h) \\ Y_1(0)A_{12}X_3(0) \\ Y_1(h)A_{12}X_3(0) + Y_1(0)A_{12}X_3(h) \\ Y_1(2h)A_{12}X_3(0) + Y_1(h)A_{12}X_3(h) + Y_1(0)A_{12}X_3(2h) \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = 2 = n_2.$$

Thus System (24) is $\mathbb{R}^n$-observable with respect to x_3.
We leave the example of the dual system to the reader.

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References

1. Balachandran, K.; Govindaraj, V.; Rivero, M.; Tenreiro Machado, J.A.; Trujillo, J.J.: Observability of nonlinear fractional dynamical systems. Abstr. Appl. Anal. Art. ID 346041, 7 pp. (2013)
2. Guermah, Said; Djennoune, Said; Bettayeb, Maamar: Controllability and observability of linear discrete-time fractional-order systems. Int. J. Appl. Math. Comput. Sci. 18, no. 2, 213—222 (2008)
3. Guo, Tian Liang: Controllability and observability of impulsive fractional linear time-invariant system. Comput. Math. Appl. 64, no. 10, 3171—3182 (2012)
4. Kilbas A.A., Srivastava H.M., Trujillo J.J.: Theory and Applications of fractional Differential Equations. 4th ed., Springer, Berlin, (2007)
5. Moubarak, M.R.A.; Rida, S.Z.; Abd-elrady, A.S.; Mohamed, A.K.: Controllability and observability for infinite dimensional fractional singular control systems. J. Fract. Calc. 38, 47—61 (2010)

6. Marchenko, V.M.; Zaczkiewicz, Z.: Duality in controllability and observability problems for nonstationary hybrid impulsive differential-difference systems. Differ. Equ. 46, no. 12, 1767—1778 (2010)
7. Marchenko, V.M.; Luazo, Zh.Zh.: On the stability of hybrid differential-difference systems. Differ. Equ. 45, no. 5, 743—756 (2009)
8. Marchenko, V.M., Poddubnaya, O.N., Zaczkiewicz, Z.: On the observability of linear differential-algebraic systems with delays. IEEE Trans. Automat. Control. 51, No. 8, 1387–1392 (2006)
9. Marchenko, V.M. and Poddubnaya, O.N.: Relative controllability of stationary hybrid systems. in Proc. IEEE Methods and Models in Automation and Robotics (MMAR 2004), Miedzyzdroje, Poland, Aug./ Sep., 267—272 (2004)
10. Sabatier, Jocelyn; Farges, Christophe; Merveillaut, Mathieu; Feneteau, Ludovic: On observability and pseudo state estimation of fractional order systems. Eur. J. Control 18, no. 3, 260—271 (2012)
11. Zaczkiewicz, Z.: Controllability, observability, duality for fractional differential-algebraic systems with delay. Journal of Applied Nonlinear Dynamics, Nr 1, 195–205 (2012)
12. Zaczkiewicz, Z.: Representation of solutions for fractional differential-algebraic systems with delays. Bull. Pol. Acad. Sci. Tech. Sci., Vol. 58, nr 4, 607–612 (2010)
13. Zaczkiewicz, Z. Observability of linear differential-algebraic systems with delay. Ph.D. Thesis. Warsaw University of Technology: Poland (2009)

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