9354

IV

111 a I

Rackham

On the Practical Applicability of Stokes' Law of Resistance, and the Modifications of it Required in Certain Cases.

by

Prof. M. S. Smoluchowski

[54]

dr. Intern. Congress of Math. Cambridge 1912
p. 10

On the Practical Applicability of Stokes' Law of Resistance, and the Modifications of it Required in Certain Cases.

By M. S. Smoluchowski, Ph.D., L.L.D., Professor of Physics at the University of Lemberg.

---

§1). Stokes' law for the resistance of a sphere in a viscous liquid rests, as is well known, on the fundamental assumptions:

I. Slowness of motion, so that the inertia terms in the hydrodynamical equations may be neglected, in comparison with the effects of viscosity,

II. Complete adhesion without slip, of the liquid to the sphere, this being considered as a rigid body,

III. Unboundedness of the liquid and immobility at infinity.

In the following I should like to contribute some remarks on this law with regard to certain cases of practical importance, where the underlying conditions are changed to some extent, which may be of some interest to those who are engaged with research work on subjects connected with Stokes' law.

First let us touch briefly the question of slipping, connected with the second of the above assumptions. Stokes' calculation can be generalised, by allowing the liquid to slip

along the surface of the sphere, with a velocity proportional to the frictional force in the tangential direction, [which in the case of a parallel laminar flow ~~assumes~~ implies the surface condition: $\beta u = \mu \frac{\partial u}{\partial y}$].

In this case, as Basset has shown, the simple law of Stokes has to be replaced by:

~~$F = 6\pi\mu R$~~

$$F = 6\pi\mu R c \, \frac{\beta R + 2\mu}{\beta R + 3\mu} \qquad \text{------ (1)}$$

Thus the minimal value of the resistance, for the case of infinite slip ($\beta = 0$), is two third of the maximal value for no slip ($\beta = \infty$).

Now it is generally assumed, on account of the experimental researches of Poiseuille, Whetham, Couette, Ladenburg and others, that the slip of liquids along solid walls is negligibly small. Mr. Arnold's recent ~~research~~ measurements prove, by their exact agreement with Stokes' law, that the coefficient of sliding friction β is certainly greater than 5.000 and probably greater than 50.000. [~~Still greater values would result from the fact that even the resistance of electrolytic ions agrees corresponds by its order of magnitude quite well to molecular diameters~~]

§2). On the other side ~~Mr. Arnold~~ his experiments, on bubbles of gas moving through liquid, gave the unexpected result that ~~the~~ the slip at clean[2] surfaces between gas and liquid is infinite, as the velocity turned out too

1) H. D. Arnold, Phil. Mag. 22 p. 755 (1911).

2) i.e. provided the surface be not contaminated with solid films.

great by 50 per cent.

Now I think a different explanation of those experiments to be preferable, as in the case of gas bubbles or liquid drops also the interior liquid is subject to circulation. Some time ago I advised Mr. Rybczynski in Lemberg to calculate the motion of a visqueous sphere through viscous liquid. The ~~exact~~ calculation is ~~[illegible] easy~~ quite easy and the result,[1] published january last year, and deduced also half a year later, quite independently of course, ~~half a year later~~ by M. Hadamard, is equally simple. It shows that for slow motion the inner liquid retains its spherical ~~sh~~ shape and that the resistance is:

$$F = 6\pi\mu R c \,\frac{3\mu' + 2\mu}{3\mu' + 3\mu} \qquad \text{---} (2)$$

where μ' designates the viscosity of the liquid in the interior of the sphere.

Comparison with the above formula shows that the resistance experienced by a gas bubble or liquid drop without slip is the same as the resistance of a solid sphere with a coefficient of surface friction $\beta = 3\frac{\mu'}{R}$; in fact the velocity and the stream lines of the outer liquid are identical in both cases. It would be interesting to verify the above formula by experiments on liquids with similar values of μ and μ'; in the case of Mr. Arnold's experiments the viscosity in the interior was negligible in comparison with ~~th~~ the

1) W. Rybczynski, Bull. Acad. d. Sciences Cracovie 1911 p. 40; J. Hadamard, Comptes Rendus 152, p. 1735 (1911); 153 p. (1912).

viscosity of the outer medium, which had the same effect as if the surface slip were infinite. So far ~~the~~ his results too are explained without the assumption of surface slip.

§3) However, there is a case where the existence of surface slip has been proved beyond doubt: in rarefied gases. As is well known, the magnitude of the coefficient of slipping $\gamma = \frac{\mu}{\beta}$ is, according to the kinetic theory and also to the old experiments of Kundt and Warburg, ~~app~~ ~~approximately~~ roughly equal to the mean length of the free path of the gas molecules; therefore the phenomenon plays an important part even at ordinary pressures in the motion of very minute droplets, as in Millikan's experiments.

Now ~~it would seem natural to~~ unfortunately one cannot use formula (1) for this case, with substitution of the empirical value for ~~γ~~ β, ~~but such a procedure would give quite erroneous results~~, except for the case of comparatively small slip. For if the mean length λ is comparable with the dimensions of the moving sphere, the ordinary hydrodynamical equations ~~[illegible]~~ cease to be valid altogether, since the implicit assumption underlying them, that the state of the gas is varying little for distances comparable with λ, is impaired.

Therefore also the interesting deduction of a corrected formula by Prof. E. Cunningham[1] is not to be considered as a demonstration and Messrs. Knudsen and S. Weber may be right in trying to get closer approximation by (other, purely) empirical

1) E. Cunningham, Proc. Roy. Soc. 83, p. 357 (1910)

formulas.[1] At any rate the formula proposed by Cunningham

$$F = 6\pi\mu Rc\left[1 + A\frac{\lambda}{R}\right]^{-1}$$

serves remarkably well for interpolation, considering the experiments of the ~~these~~ authors and those of Mr. Mc. Keehan[2]. It is preferable to write it in the form $F = 6\pi\mu Rc\left[1 + \frac{B}{R\rho}\right]^{-1}$

where ρ is the density of the gas, as mistakes are easily involved by using the mean length of free path (λ) which is a very indefinite term and really has no precise meaning.

For great rarefaction the resistance is proportional to the cross section of the sphere, and for this case the calculation can be carried out exactly, if the way is known, how the interaction between the surface of the sphere and the gas molecules takes place. If they rebound like elastic bodies, we get in accordance with Cunningham

$$F = \frac{4}{3}\sqrt{\frac{8}{3\pi}}\,R^2\pi\rho c V$$

where V is the square root of the mean square of molecular velocity. The empirical coefficient, as ~~following~~ calculated from the experiments mentioned above, is considerably larger, it amounts to 1·65 (Knudsen and Weber) or 1·84 (Mc. Keehan). Mc. Keehan concludes that molecules are reflected from the surface of the sphere only in a normal direction; I think however (that) his theoretical formula is not quite exact

1) M. Knudsen u. S. Weber, Ann. d. Phys. 36 p. 981 (1911).

2) Mc. Keehan, Physik. Zeitsch. 12, p. 707 (1911).

and at any rate his conclusion seems to me at variance with fundamental principles of the kinetic theory of gases. I think that the experimental results are explained best by the view, supported also by other researches of this kind, especially those of Knudsen, that a solid surface acts in scattering the impinging molecules irregularly in all directions, whether with or without change of mean kinetic energy. We shall not go into these questions now, however, as they belong to the kinetic theory of gases, not to hydrodynamics.

§4). Now let us consider what modifications are required in Stokes' law, if the third of the fundamental assumptions is impaired, the liquid being limited by solid walls, or a greater number of similar spherical bodies being contained in it.

In this case the linear form of the hydrodynamical equations makes it possible to attain their solution by a method of successive approximations, analogous to the methods of images used in the theory of electrostatic potential. It consists in the successive superposition of solutions formed as if the fluid would extend to infinity, but so chosen as to annull the residual motion at the boundaries, with increasing approximation.

This method was used first by H. Lorentz in order to determine the influence of an infinite plane wall on the progressive movement of a sphere, and we shall refer to his formulas later on.[1] He found that the resistance of the sphere is increased by a fraction amounting to $\frac{9}{8}\frac{R}{a}$ for normal motion, $\frac{9}{16}\frac{R}{a}$ for parallel motion, if a denotes the distance from the wall. Mr. Stock in Lemberg has extended the calculation for the second

[1]) H. A. Lorentz, Abhandlungen ü. th. Physik I p. 23 (1906). In Millikan's determinations of the ionic charge the increase of resistance due to the presence of the condenser plates, may produce an increase of the order of one thousandth.

case to the fourth order of approximation, {including terms with $\left(\frac{R}{a}\right)^4$}. [1]

In a somewhat similar way Ladenburg [2] calculated the resistance experienced by a sphere, when moving along the axis of an unlimited cylindrical tube, and his result, indicating an increase in comparison with the usual formula of Stokes in the proportion of $1 : 1 + 2{\cdot}4\,\frac{R}{\rho}$, (where ρ = radius of the tube), has been verified with very satisfactory approximation by his own experiments and by those of Mr. Arnold.

§5). Now let us apply this method to the case, where a greater number of similar spheres are in motion, and extend a little further now an investigation which I had begun in a paper published last year. [3] Imagine a sphere of radius R, moving with the velocity c along the X axis, its centre being situated at the distance x from the origin. It would produce at the point P (with coordinates ξ, η, ζ) certain current velocities $u_0\ v_0\ w_0$, of order $\frac{Rc}{r}$, defined by Stokes' equations, if the fluid be unlimited. But if we assume this point P to be the centre of a solid sphere of radius b, we have to superpose a fluid motion $u_1\ v_1\ w_1$, chosen so as to annull the velocities of the primary motion at the points of this sphere and satisfying the conditions of rest for infinity.

1) J. Stock, Bull. Acad. Scienc. Cracovie 1911 p. 18.

2) R. Ladenburg, Ann. d. Phys. 23, p. 447 (1907).

3) M. Smoluchowski, Bull. Acad. Scienc. Cracovie 1911 p. 28

This motion may be called the „reflected" motion; it can be found with any degree of approximation, by making use of the solution of the hydrodynamical equations given by Lamb, in form of a development in spherical harmonics. But as it is of order $\frac{Rc}{r}$ at the surface of the second sphere, which is its origin, it seems probable a priori, that its magnitude at the first sphere will be of order $c\left(\frac{R}{r}\right)^2$, and I have verified this as well as the following results by explicit calculation. Thus if we confine ourselves to terms of order $c\left(\frac{R}{r}\right)^2$, we can apply a simplified method of evaluating the mutual influence of such spheres, by neglecting the difference between the velocity at the centre of the second sphere and at its surface.

That is to say: the sphere P, being at rest, is subjected to frictional forces:

$$X = 6\pi\mu R u_0$$

$$Y = 6\pi\mu R v_0$$

$$Z = 6\pi\mu R w_0$$

on account of the motion of the first sphere; on the other side, the moving sphere experiences a reaction by virtue of the presence of the sphere P, such as if this would execute simultaneously the three motions $-u_0, -v_0, -w_0$; the three current systems resulting therefrom, according to the usual formulas of Stokes, produce at the centre of the first sphere nine current components, giving rise to nine components of frictional force, to be calculated each according to Stokes' law of resistance.

If both spheres are in simultaneous motion, the mechanical effects ~~result~~ are found by superposition of the forces corresponding to the two cases where one of them

is moving and the other one at rest.

In this way an interesting conclusion is obtained for the case where both spheres are moving in parallel directions with equal velocity: then both are subjected to equal additional forces in the same direction, one component in the direction of motion, tending to diminish the resistance by the amount: $\frac{9}{2}\frac{R^2\pi\mu c}{r}\left[1-\frac{3}{4}\frac{R}{r}\right]$, the other component along the line joining the centres, towards the sphere which is going ahead, of amount: $\frac{9}{2}\frac{R^2\pi\mu c\cos\theta}{r}\left[1-\frac{9}{4}\frac{R}{r}\right]$.

[where θ is the angle between the line of centres and the direction of motion].

Thus two heavy spheres of this kind would ~~sink~~ sink faster than Stokes' law is indicating and besides, their path must be deflected from the vertical towards the line of centres by an angle ε defined by:

$$\sin\varepsilon = \frac{3}{4}\frac{R}{r}\left[1-\frac{3}{2}\frac{R}{r}\right]\sin\theta\cos\theta$$

§6). Analogous methods are applicable to a greater assemblage of spheres. The motion results from superposition of simpler solutions, where one sphere is supposed moving and all the other ones ~~at~~ resting. Each of the component solutions ~~the~~ comprises the direct action, and for higher approximation also its „reflections."

Now if the parallel motion of a cloud of n similar spheres ~~fro~~ is considered, the resistance of each of them will be diminished by an expression ~~at~~ proceeding after powers of R, the first term of which will be of the order of magnitude $\mu c R^2 \sum \frac{1}{r}$. We see that these developments would be divergent for an infinite number of spheres. It is evident that for instance an infinite row of spherical particles, arranged at equal distances, would aquire infinite velocity, by virtue of their gravity, as also an infinite cylinder would behave in the same way. This applies a fortiori to two dimensional infinite assemblages.

Thus Stokes' law of resistance will not be true even approximately, and the development will cease to be convergent in general, unless ~~[struck out]~~ $\frac{nR}{S}$ is small, where S denotes a kind of mean distance, ~~[struck out]~~ comparable with the linear dimensions of the cloud.

§7. The same result follows from the following simple reasoning. Imagine a spherical cloud of radius S, containing n spherical particles, each of radius R and density σ, suspended in a medium of viscosity μ, of negligible density, for example a cloud of minute drops of water in air. Then currents will take place in the spherical cloud and it will attain a certain velocity as a whole, which may be calculated after the formula (2), just as if the cloud would form a homogeneous medium of density ~~[struck out]~~ $n\left(\frac{R}{S}\right)^3$ and of the same viscosity as the outer medium. The mass velocity resulting therefrom, of amount $\frac{4}{15}\frac{nR^3g\sigma}{S\mu}$ is superposed upon the displacement of the particles, relative to the moving medium, taking place with velocity $\frac{2}{9}\frac{R^2g\sigma}{\mu}$. Thus evidently the downward velocity will be much increased, and Stokes' law cannot be true even approximately, unless $\frac{nR}{S}$ is small in comparison to unity. This condition shows that Stokes' law can be applied only to particles constituting clouds of exceedingly scarce crowding, and it is easily seen that it would be quite erroneous to apply it to actual fogs, (or actual clouds in the atmosphere,) with diminished transparency [as in this case the aggregate cross-section of the particles $nR^2\pi$ is comparable with the cross-section of the cloud $S^2\pi$]. As an illustration how cautious we must be in this respect, I may mention that the ratio $\frac{nR}{S}$ amounts to 10 and even to 100, for a cubic centimetre cloud

as produced by J. J. Thomson and H. A. Wilson, in their experiments ~~#~~ on the determination of the ionic charge.

§8). What has been said, applies of course only to clouds moving in an otherwise unlimited medium. The conditions of motion are quite different for a cloud contained in a closed vessel (as in the experiments just referred to). Prof. E. Cunningham has attempted to evaluate the order of magnitude of the correction ~~to of Stokes' law~~, to be applied to Stokes' law in this case. His estimate is founded on the supposition that each particle moves approximately in such a way, as if it were contained in a rigid spherical envelope, of radius comparable with ~~the~~ half the distance to its next neighbours. Now ~~this~~ supposition does not seem quite evident, although we shall see that it leads to a result of the right order.

~~The~~ We can calculate the resultant motion in quite an exact way, if we consider a homogeneous assemblage of equal spherical particles, moving all of them with the same velocity c in the direction of negative X, toward an infinite rigid wall, which we assume to be the plane YZ. In this case we see, by making use of H. A. Lorentz's calculation before alluded to, that a moving sphere x, y, z produces at a point ξ, situated on the axis of X, a velocity component

$$u = -\frac{3}{4}\frac{Rc}{r}\left[1 + \left(\frac{\xi - x}{r}\right)^2\right] + \frac{3}{4}\frac{Rc}{\rho}\left[1 + \frac{x^2 + \xi^2}{\rho^2} + \frac{6x\xi(x+\xi)^2}{\rho^4}\right] \quad \ldots\ldots (3)$$

The first part of this expression, containing $r = \sqrt{(x-\xi)^2 + y^2 + z^2}$, is the component of direct motion, according to Stokes; the second part is the component caused by „reflection" at the plane YZ; it contains the distance between the point ξ and

the reflected source $\rho = \sqrt{(x+\xi)^2 + y^2 + z^2}$.

The terms with higher powers of $\frac{R}{r}$ have been neglected, as we confine ourselves to the first approximation. The total current produced in the point ξ by the motion of all the particles is equal to: $U = \sum u$, where the summation is to be extended over all their values of x, y, z. Now we might think us ~~quite~~ entitled to replace the summation by an integration, considering that one particle corresponds to the space ~~[illegible]~~ A^3, if A denotes ~~[illegible]~~ a sort of mean distance between the particles. In this case the result would be very simple, for we should have:

$$U = \frac{1}{A^3} \iiint u \, dx\, dy\, dz$$

~~[illegible]~~ The integrals of the separate terms constituting u can be evaluated explicitly, if we extend them to a cylinder with $V2$ as basis, of height h and of radius S. Then we can use the well known expression for the potential of a disk in points of its axis, and expressions derivable from it by differentiation with respect to ξ, and by these means we find the unexpected result that the integral current U is zero, if we extend the summation to an infinite value of S.

But in reality U is not defined by integration but by summation. Evidently both operations lead to the same result for distant parts of the space, but not for those parts whose distance from the point ξ is comparable with the distances A between two particles. Therefore the resultant current U in points at a great distance (in comparison with $\frac{A}{2}$) from the wall will be given by:

$$U = \frac{3}{4}\frac{Rc}{A}\beta$$

where $\beta = \frac{1}{A^2}\iiint \frac{1}{r}\left(1+\frac{x^2}{r^2}\right) dx\,dy\,dz - \sum \frac{A}{r}\left(1+\frac{x^2}{r^2}\right)$ $\Big\}\ldots(4)$

to be extended over a space great in comparison with A, is a purely numerical coefficient. ~~This~~

In order to evaluate β we must know how the particles are arranged. If we suppose ~~for instance~~ an arrangement in ~~cubic~~ rectangular order, we can get easily an approximate value by explicit calculation and by integrating over a cube of height H, constructed around the point ξ, which gives

$$\iiint \frac{1}{r}\left(1+\frac{x^2}{r^2}\right) dx\,dy\,dz = 8\,H^2\left[\log\left(1+\sqrt{3}\right) - \frac{1}{2}\log 2 - \frac{\pi}{12}\right]$$

It is sufficient to take H equal to a small uneven multiple of $\frac{A}{2}$, as the expression for β is rapidly converging with extension of the limits of integration. In this way I have found the approximate value $\beta = 3{\cdot}09$, and therefore the resistance for one particle will be

$$F = 6\pi\mu Rc\left[1+\frac{3}{4}\frac{R\beta}{A}\right] = 6\pi\mu Rc\left[1+2{\cdot}32\frac{R}{A}\right] \quad \text{------(5)}$$

This formula would apply, of course also if the particles were arranged in a different way, but then the numerical value of β would ~~[illegible]~~ be different. Our result agrees to the order of magnitude ~~[illegible]~~ with ~~[illegible]~~ of Prof. Cunningham's estimate, which ~~[illegible]~~ lead him for the case of an equilateral arrangement to a similar formula, with a coefficient of $\frac{R}{A}$ included within the limits $3{\cdot}67$ and $4{\cdot}5$.

§ 9). However, the practical application of this formula is rather questionable, as it applies only to a regular arrangement of particles. If they were arranged in clusters,

the correction might even become negative. It is interesting to note that the average value of β, for a particle whose position relatively to the other ones is defined by pure accident, should be zero. That seems quite natural and, as the average current of liquid U in the cross section must be zero. Thus it follows what we should not have expected at first sight, that ~~to this order of approximation~~ Stokes's law ~~would~~ applies for the particles of an actual cloud, on an average with no correction whatever, of this order of magnitude.

The evaluation of the quadratic terms would be much more complicated of course, ~~as not only the reaction of the~~ as then all possible kinds of single reflections caused by any one sphere, have to be taken into account.

The general result of our calculation shows at any rate that Stokes's law is undergoing but small corrections, if applied to the particles of a uniform cloud filling a closed vessel. But it is important to note that things will change entirely, if the cloud is not of quite uniform density, or if it does not fill the whole empty space between the walls. Then as a rule convective currents will arise, ~~the velocity of~~ which in certain cases may be of preponderant influence. Their velocity may be calculated approximately, by considering the medium as a homogeneous liquid, subjected to certain forces, the intensity of which per unit volume corresponds to the aggregate force acting on the particles contained in it.

Consider for instance an electrolyte in an electric field. If it is conducting in accordance with Ohm's law, the average electric density is zero and no currents will take place. But in bad liquid conductors, with deviations from Boyle's law

convective currents may arise, which may influence also materially the apparent value of the conductivity. They have been observed long ago, for instance by Warburg[1]

Similar movements may be produced in ionised gases, and I think more attention ought to be paid to them ~~possible deviations from the normal ionic mobility, connected with Stokes' law, in this case~~ than usually is done. In experiments where the saturation current of strong radioactive material is observed between condenser plates wide apart[2], these phenomena may be of importance as producing an apparently greater mobility of the ions than under normal conditions.

§ 10). There is another ~~one more~~ application of the theoretical methods exposed above, which may be mentioned. Imagine a two dimensional infinite assemblage of equal spherical particles, distributed ~~arranged~~ uniformly over the plane $x = l$, whilst the plane YZ again may be supposed to be a rigid wall. Now let all these particles be moving along the plane, in direction Y with ~~the~~ equal velocity c; what motion will be produced in the surrounding liquid, what will be the resistance experienced by every particle?

According to Lorentz again the motion produced by a single sphere ~~at~~ moving parallel to a fixed wall is, with neglection of higher powers of the ratio $\frac{R}{l}$, which we suppose to be a small quantity:

$$v = \frac{3}{4}\frac{Rc}{r}\left[1+\left(\frac{y}{r}\right)^2\right] - \frac{3}{4}\frac{Rc}{\rho}\left[1+\left(\frac{y}{\rho}\right)^2\right] - \frac{3}{2}\frac{Rcx(x+\xi)}{\rho^3} + \frac{9}{2}\frac{Rcxy^2(x+\xi)}{\rho^5}$$

1) E. Warburg. Wied. Ann. d. Phys. . .

2) F. inst.: Rutherford, Radioactivity p. 35, p. 84.

where the first term is the direct current according to Stokes, while the remaining terms represent the ~~reflected~~ current reflected by the wall, just as in the former example.

We might also in this case calculate the resultant current ~~z~~ by forming $\sum v$ over all values of y and z, and derive therefrom the resistance of a single particle. But we shall confine ourselves to the following remarks.

In the extreme case where the particles are so crowded, as to nearly touch one another, a lamellar flow will take place in the liquid, between the fixed wall and the plane $x=l$, with a velocity $v = \frac{cx}{l}$, while on the other side of the plane $x=l$ the liquid will be dragged along by the sheet of moving particles with the constant velocity c. The frictional force per unit of surface of the plane $x=l$ is evidently equal to $\frac{\mu c}{l}$, therefore the resistance experienced by each particle is

$$F = \frac{\mu c A^2}{l}$$

which is much smaller than Stokes' law would indicate, as A is of the order of R_F but the distance l is supposed to be of higher order.

Now consider the other extreme case, where the distances A between the particles are so great, that Stokes' law is approximately valid, which requires A to be of order l. Let us calculate the resultant motion of the liquid, for ~~p~~ points at infinite distance from the wall ~~...~~ ($\xi = \infty$). For such points the summation mentioned above can be replaced by integration; besides we can put: $\frac{1}{r} - \frac{1}{\rho} = \frac{2l\xi}{r^3}$, $\frac{1}{r^3} - \frac{1}{\rho^3} = \frac{6l\xi}{r^5}$; and thus we get

$$V_\infty = \sum v = \frac{9Rcl\xi}{A^2} \iint \frac{y^2\,dy\,dz}{(\xi^2+y^2+z^2)^{5/2}}$$

This integral can be transformed by putting: $y = s \sin\varphi$, $z = s\cos\varphi$, $dy\,dz = s\,ds\,d\varphi$, and we get finally:

$$V_\infty = \frac{6 R l \pi c}{A^2}$$

By comparing this with Stokes' law for the resistance F we have

$$V_\infty = \frac{F}{A^2}\,\frac{l}{\mu}$$

that means that in both cases the liquid at a great distance from the wall will be dragged along, in a parallel direction to it, with such a velocity as if the force corresponding to unit surface $\frac{F}{A^2}$ were distributed uniformly over the liquid, in a plane at a distance l from the fixed wall. This result, which can be generalised for a greater number of similar layers, seems natural enough, if the distances between the particles are small in comparison with their distance from the wall, so that the assemblage can be considered as if forming a homogeneous medium, but we see it remains true for particles widely apart. Without going into further details, I may only mention that this result has an important bearing on the theory of electric endosmose, which will be explained elsewhere with full details.

§11). I may conclude with a brief remark about the influence of the inertia terms in the hydrodynamical equations (assumption I), which have been neglected as well in Stokes' original calculations as in the above reasonings. It is well known that this neglection is justified only, if the ratio $\frac{Rc\sigma}{\mu}$ is small in comparison to unity. But it has been proved by Oseen[1)] in an important paper, commented in a very interesting way upon by Lamb, that the solution given by Stokes is defective, even if this

1) Oseen, Arkiv f. mat. astr. fysik 6 (1911); H. Lamb, Phil. Mag. 46 p. (1911)

criterion is fulfilled; for at distances r where $\frac{rc\sigma}{\mu}$ is large, the inertia terms must be of prevalent influence over viscosity. Oseen himself has given a solution which is different from Stokes' equations ~~which gives better approximation~~ for ~~that~~ those distant parts of the space and gives better approximation there. However, the resistance of the sphere ~~in a liquid of finite extent~~ depends only on the state of movement in its immediate neighbourhood, therefore the resistance law of Stokes is not impaired by those results. The condition of its validity may be defined ~~still~~ more exactly by means of the recent experiments of Mr. Arnold, which have shown that it ~~the~~ holds with very good accuracy for spheres ~~falling~~ moving under influence of gravity, provided their radius is smaller than $0{\cdot}6\,\bar{r}$, where the critical radius $\bar{r}$ is defined by the relation $\frac{\bar{r}\,\bar{v}\,\sigma}{\mu} = 1$. This means ~~[illegible]~~ that the ratio $\frac{Rc\sigma}{\mu}$ must be smaller than $(0{\cdot}6)^3 = 0{\cdot}22$.

§12). The inertia terms are of greater importance, ~~as modifying the law~~ in the case before alluded to, where the motion of a greater number of similar spheres is considered. For it is legitimate to calculate the forces of reaction between such spheres ~~only~~ by using ~~the~~ Stokes' equations for slow motion, only if they are lying within the space where viscosity is predominant over inertia. Mr. Oseen has generalised recently[1] the calculation of the interaction of two spheres given by me, by introducing in it his solution of Stokes' problem. The forces excited on the two spheres come out unequal in this case and are given by much more complicated expressions. They become identical with the first approximation

1) F. Oseen, Arkiv f. mat. astr. fysik 7 (1912)

given by me, if the distance r between the two spheres satisfies the condition that $\frac{rc\delta}{2\mu}$ is small. Mr. Oseen thinks this to be a great restriction on the validity of those formulas for experimental purposes, but he ~~[illegible]~~ omits the factor δ in the above expression. We satisfy ourselves easily that for instance in the case of waterdrops in air, as in ~~[illegible]~~ Sir J. J. Thomson's and H. A. Wilson's condensation experiments, r is of the order of several centimeters; in Perrin's experiments on the validity of Stokes' law for the particles of emulsions it would amount to hundreds of meters. It is also sufficiently great for direct experiments, when highly viscous liquids are used, as Ladenburg did in his elaborate research. Ordinary hydraulic experiments, with water and spheres of a size to be handled conveniently, are excluded of course when Stokes law or any of those modifications are in question.

One might try to apply Oseen's method of approximate correction for inertia ~~might be applied~~ also to the other cases treated above, but it will imply rather cumbersome calculations and besides, for movements in closed vessels it will be generally of lesser importance than in a liquid extending to infinity.

~~Some~~ Remarks on Stokes's Law and Some Modifications of it.

~~On the Practical Applicability of Stokes' Law.~~

On the practical applicability of Stokes' Law of Resistance and ~~on~~ the ~~corrections~~ modifications of it required in certain cases.

~~On Surface Slip and the Forces exerted between~~

~~I. Surface Slip~~ Stokes's Law for the resistance of a sphere in a viscous ~~medium~~ liquid fluid has ~~recently~~ acquired in recent times greater importance for modern physics, than any other result of theoretical hydrodynamics, especially by its applications to the determination of the electronic charge and to the theory of Brownian movements.

~~But the limits of its applicability are not always ~~clearly~~ realised; they are~~

Therefore ~~the question of the limits of its applicability and of the corrections to be introduced in certain cases is a very actual one and I should like to add some contributions to these developments.~~ merits careful study; may be allowed; theoretical; this subject

The calculation of Stokes ~~law~~ rests, as is well known, on the fundamental assumptions in the hydrodynamical equations

I. That the motion is so slow that the inertia terms may be neglected

II. That the liquid is adhering completely, without slipping, to the surface of the sphere, this being considered as a rigid body.

III. That the liquid is unlimited and at rest at infinity.

The first condition ~~restricts~~ imposes the most important restriction on the validity of S— when applied to ordinary hydraulic experiments as it requires the radius of the sphere being small in comparison to the value $\overline{r} = \frac{\mu}{v\rho}$, where μ is the viscosity, ρ the density of the liquid, v the velocity of the sphere. ~~Recent~~ The careful experimental study of Dr Arnold, published recently in the Phil Mag, shows that the law of resistance is applicable provided the radius be ~~less of~~ $r = 0{\cdot}6\,\overline{r}$ ~~but Ladenburg noticed~~ has been noticed ~~the~~ which would imply for v

* Phil. Mag. 21, 357, 1911

* Bull. Ac. Crac. 1911 p. 40 (January)

* C. R. 152, p. 1735 (1911)

~~The~~ Now as the critical radius for a water drop in air moving under influence of its gravity is $\overline{r} = 5 \cdot 10^{-3}$ cm we see that this condition is certainly satisfied for the minute droplets of diam (-) which ~~condensation experiments of Thomson Wilson experiments and by~~ Millikan Thomson, Wilson, Millikan ~~and others~~ and to used by determine the ionic charge, and ~~like~~ likewise it was fulfilled for Perrin's ~~condensed~~ experiments on ~~the movement of the~~ suspensions of particles of a gum gutt and mastix emulsion in water ~~suspension~~.

~~The~~ On the other side ~~Oseen~~ it has been proved by Oseen in a very interesting paper that the hydrodynamical solution ~~that~~ given by Stokes is defective ~~for~~ for spheres of any diameter if the motion is considered which takes place ~~outside~~ at distances exceeding the critical radius is considered in parts of the space where the inertia terms are of prevalent influence.

~~Oseen himself~~ ~~[illegible]~~ However, Stokes' law of resistance remains ~~approximately~~ valid, under the above conditions and it is not necessary in so far ~~as resistance~~ to replace the simple solution of Stokes by the rather ingenious but complicated method calculation, indicated by Oseen and Lamb, which give a better approximation of the hydrodynamical problem for points at great distances from the sphere.

Bosset

Loub dez 2

* Phil Mag. 22, 755, 1911

* Bull Ac. Crac. 1911 p. 40 (Janvier)

* C.R. 152, p. 1735 (1911)

~~In the following~~ [~~I shall not go into further consideration of the influence of inertia terms, but~~] First I should like to call attention ~~the first~~ to the second of the above named conditions.

~~If instead of the~~ It is easy to generalise Stokes calculation by introducing (the general supposition that the liquid surface layer of instead of being at rest relatively to ~~the surface of~~ the sphere, is moving along its surface with a tangential velocity proportional to the frictional force, $v = \beta F$ [which in the case of a parallel lamellar flow assumes the form $\beta u = \mu \frac{\partial u}{\partial y}$]

In this case as (Stokes and) Basset have shown, the simple law of Stokes has to be replaced by

$$F = 6\pi\mu a c \frac{\beta a + 2\mu}{\beta a + 3\mu}$$

~~so the~~ Thus the frictional resistance would be diminished by surface slip and ~~slipping~~ the minimal value, for the case of infinite slip, $\beta = 0$, is two thirds of the maximal value, for no slip (infinite exterior surface friction).

Now it is generally assumed that the slip of liquid at solid walls is negligibly small; on account of the researches of Poiseuille, Whetham, Couette, Ladenburg; Mr. Arnold's recent research proves, by the exact validity of Stokes law, that the coefficient of sliding friction β is certainly greater than 5000 and probably greater than 50.000. ~~[illegible]~~, I think it even probable that the coefficient β may be of the order 10^6 Still greater values would result from the fact that even the resistance experienced by electrolytic ions agrees in its order of magnitude with Stokes law.

On the other side Mr. Arnold concluded from his results that the slip at the clean surfaces between gas and liquid for bubbles of gas moving through liquid is infinite, (i.e.) provided the surface be not contaminated with solid films that were observed [~~except in the case where solid films are formed on the surface~~]

Now I think a different interpretation explanation of Dr. Arnold's experiments preferable, as in the case of gas bubbles or liquid drops also the interior fluid is subject to circulation.

Some time ago I advised Mr. Rybczyński in Lemberg to calculate the motion of a viscous sphere through viscous liquid. (The result published January last year and ~~which has been~~ The calculation is surprisingly easy.) deduced also half a year later quite independently by M. Hadamard, is equally simple. It shows that in the case of slow motion the inner liquid retains of course its spherical shape and that the resistance is

$$F = 6\pi\mu a c \frac{3\mu' + 2\mu}{3\mu' + 3\mu}$$

where μ' designates the viscosity of the liquid of which the moving sphere is composed.

Comparison with the above formula shows that the resistance experienced by a gas bubble or liquid drop without surface slip is the same as the resistance of a solid sphere with a coefficient of surface friction $\beta = \frac{3\mu'}{a}$; in fact the stream lines and velocity of the outer liquid are identical in the two cases. In the case of Dr. Arnold's experiments the coefficient μ' was negligibly small in comparison with μ which had the same effect as if there were an infinite slip at the surface. It would be interesting to verify the ~~exactness of~~ formula

( )

In the following I should like to add some contributions to the discussion of its validity and of the corrections to be applied for practical purposes and besides to touch some interesting hydrodynamical problems connected with this subject.

by experiments on liquid drops moving slowly in a liquid medium of similar [illegible] viscosity; [illegible]

Thus also Dr. Arnold's results are explained without the assumption of surface slip.

However the existence of surface slip has been proved beyond doubt in the case of rarefied gases, by direct experiments and where its order of magnitude has been explained by theory:

As is well known the magnitude of the coefficient of slip $\gamma = \frac{\mu}{\beta}$ is, according to the kinetic theory and also to the old experiments of K. & W., approximately equal to the mean length of the free path of the gas molecules; therefore the phenomenon of slip plays an important part even at ordinary pressures in the motion of drops of exceedingly minute dimensions as for instance in Millikan's and in the [illegible] of Mr. Kelvin's experiments.

Now Dr. Cunningham has undertaken to deduce by aid of the kinetic theory a theoretical formula for the resistance

Now it would seems natural to use formula ( ) for this case, with substitution of the empirical value for γ, but such a procedure would give quite erroneous results, except for the case of a comparatively small slipping factor. For if the mean length λ is comparable with the dimensions of the moving sphere the application of the usual hydrodynamical equations for viscous motion is inadmissible altogether, as the implicit assumption underlying them, that the state of the gas is varying little for distances comparable with λ, is impaired.

Dr. E. Cunningham has undertaken to deduce by aid of the kinetic theory a theoretical formula for the resistance in this case. The general form of his formula $F = \frac{6\pi\mu a c}{\left[1 + A\frac{\lambda}{a}\right]}$, [where A is a numerical coefficient depending on the way how the gas molecules behave at their impact on the surface of the sphere, and on the way how the mean length of free path λ is defined] is in comparatively good agreement [illegible]. This theoretical deduction is not really a demonstration and I think Mr. Knudsen is right in trying to get closer approximation by empirical formulas. At any rate the formula proposed by Cunningham $F = \frac{6\pi\mu a c}{1 + A\frac{\lambda}{a}}$ is a very simple interpolation formula which agrees comparatively well with the exp. of [illegible] and Knudsen.

But it is certainly preferable to write it in the form $F = 6\pi\mu a c \left[1 + \frac{B}{a\rho}\right]^{-1}$ where ρ is the density of the gas, as mistakes are easily involved by using the mean length of free path λ, which is a very indefinite term and really has no precise meaning. For great rarefaction the resistance is proportional to the cross section of the sphere* and for this case the calculation can be carried out quite exactly if the way is

* This was [illegible] [illegible] Langevin [illegible] and [illegible]

Arkiv f. mat. astr. o. fys. 6,
Lamb Phil. Mag

Arkiv .... 7 № 33 (1912)

Now let us examine what modifications are required in Stokes' law, if the third of the above named conditions is impaired, that is, if the viscous medium is limited by solid walls, or if a greater number of similar spherical bodies are contained in it.

In this case the linear form of the hydrodynamical equations makes it possible to attain their solution by a method of successive approximations, analogous to those used in the theory of electrostatic potential.

It consists in the alternate superposition of solutions chosen as to destroy the residual motion at those parts of the boundaries which are represented by solid walls, formed as if the fluid would extend to infinity.

In this way H. A. Lorentz has calculated the correction of the first order to be applied to the resistance of a sphere progressing in the neighbourhood of an infinite plane wall in a normal or in a parallel direction with to it. It amounts to the fraction $\frac{9}{8}\frac{R}{a}$ for normal, $\frac{9}{16}\frac{R}{a}$ for parallel motion.

Mr. Stock in Lemberg has extended the calculation for the second case to the fourth order of approximation (the fourth power of the ratio $\frac{R}{a}$, where R is the radius of the sphere, a its distance from the wall)

His result is
$$F = 6\pi\mu R u \left(\frac{1}{1-\frac{9}{16}\frac{R}{a}} - \left(\frac{R}{2a}\right)^3\left(1+\frac{9R}{16a}\right)\right)$$

In a somewhat similar way it has been found by Ladenburg that for the case of a sphere moving along the axis of an unlimited cylindrical tube of radius ρ the resistance calculated after Stokes increased in the ratio

$$1 : 1 + 2{\cdot}4\,\frac{R}{\rho}$$

and by superimposing the correction deduced by H. A. Lorentz also the influence of the fixed bottom of the tube can be taken into account.

Ladenburg's improved his formula so as taking into account both the influence of the cylindrical walls

and this formula has been verified with very quite satisfactory approximation by his own the very elaborate experiments of Ladenburg's own experiments and by those of Arnold.

In a paper published last year I have pointed to some interesting results, concerning the motion of a greater number of similar spheres and I may be allowed to extend these investigations a little further now.

Imagine a sphere of radius a moving with the velocity c in the direction of X its centre being situated at the distance x on this axis from the origin. If of an unlimited fluid — — — — — This would produce at the point P with coordinates $\xi\ \eta\ \zeta$ certain current velocities $u_0\ v_0\ w_0$ of the order $\frac{ac}{x}$ (defined by the well known Stokes' equations. But if we assume this point to be the centre of a solid sphere of radius b, we have to superpose a fluid motion u_1, v_1, w_1, satisfying the condition of rest for infinity and chosen so as to annull the velocities of the primary motion at the points of this sphere.

This led us to examine the modifications [illegible] required [illegible] of the [illegible] viscous motion

limited by the walls [illegible] as if a greater number of smaller spherical balls [illegible]

In this case the linear form of the hydrodynamical equations makes it possible to obtain

the solution by a method of successive approximation, consisting ~~in superposition of~~

analogous to those in the theory of electrostatic [illegible]

and I have verified this as well as the following results by explicit calculation.

It consists in the superposition of [illegible] solutions [illegible] to satisfy the conditions at the [illegible]

of the boundaries represented by solid walls, [illegible]

In this way H. A. Lorentz has calculated the correction of the first order to be applied to

the resistance of a sphere progressing in a normal or in a parallel direction to it.

[illegible]

[illegible] has extended the calculation to the fourth order of approximation of the powers

of the ratio [illegible] where R is the radius of the sphere and [illegible] its distance from the wall.

His result is $F = 6\pi\mu R v \left(\frac{1}{1 - \frac{9}{16}\frac{R}{l}} - \left(\frac{R}{2l}\right)^3 \left(1 + \frac{9R}{16 l}\right) \right)$

~~[illegible]~~

[illegible] by [illegible] that in the case of a sphere moving along

[illegible]

[illegible]

[illegible] and by [illegible]

[illegible] interesting results concerning the motion of a greater

number of smaller spheres [illegible] and I [illegible]

[illegible]

Imagine a sphere of radius a [illegible]

[illegible] from the origin. [illegible]

[illegible] velocities [illegible]

[illegible] equations. [illegible]

to [illegible] a fluid [illegible]

[illegible] so as to [illegible] the velocity of the primary motion at the [illegible] of this sphere.

This motion may be called the motion reflected; it can be found with any degree of approximation by making use of the solution of the hydrodynamical equations given by Lamb in form of a development in spherical harmonics.

~~For points at a greater distance of P it is evident~~

~~Evidently~~ it is of the order of magnitude of $\frac{ca}{R}$ at the surface of the second sphere b, which is its source? distance of origin, it seems probable a priori that it (of the order $\frac{ca^2}{R^2}$) its magnitude at the first sphere a would be.

Thus if we neglect or confine ourselves to (terms of order $c\left(\frac{a}{R}\right)^2$), we can apply a simple method of evaluating the mutual influence of such spheres, by neglecting the difference between the values of the velocity at the center of the second sphere b and at the points of its surface.

That is to say ;

the moving sphere experiences a reaction by virtue of the presence of the sphere b, (as if the sphere b would execute simultaneously the three such motions $-u_0, -v_0, -w_0$; the three current systems resulting therefrom (according to the usual formulas of Stokes) produce at the centre of the first sphere nine current components and give rise to nine components of frictional force, to be calculated each according to Stokes' law of resistance.

the sphere b being at rest, is subjected to frictional forces

$X = 6\pi\mu a b u_0$

$Y = 6\pi\mu a v_0$

$Z = 6\pi\mu a w_0$

on account of the motion of a, on the other side

If both spheres are in simultaneous motion the mechanical effects result by superposition of the forces corresponding to the two cases where one of them is moving and the other at rest.

In this way an interesting conclusion is obtained for the case where both spheres are moving in parallel lines with equal velocity: then both are subjected to additional equal forces: one component in the direction (of motion tending to diminish the resistance) by the amount $\frac{9}{2}\frac{a^2\pi\mu c}{R}\left[1-\frac{3}{4}\frac{a}{R}\right]$ in the same direction

the other component along the radius vector line from the back sphere to the front sphere centre of the sphere which follows to the sphere which goes ahead

of amount : $\frac{9}{2}\frac{a^2\pi\mu c\cos\theta}{R}\left[1-\frac{9}{4}\frac{a}{R}\right]$

(where θ is the angle between the line of centres and the direction of motion)

Analogous methods are to be applied for a greater number of spheres. The motion results from the superposition of simpler solutions where one sphere is supposed moving and all the other at rest.

Each of the component solutions results from the direct action and its reflections at the spheres.

* ~~*It*~~ (~~This~~ this increase of resistance ~~ought to be~~ is taken into account ~~also~~ for Rollikers determinations of the ionic charge;
it may produce an increase of the order of one thousandth.

* Abh. Th. Ph. I p. 23 (1906)

* Bull. Ac. Cr. 1911 p. 18 (1911)

* ~~Ann.~~ Ann. d. Ph. 23, 447 (1907)

$X = 6\pi\mu a u_0$

$Y = 6\pi\mu a v_0$

$Z = 6\pi\mu a w_0$

* Dissertat. Leipzig 1906 (Rath)

* Bull. Ac. Cr. 1911 p. 28

Now if the parallel motion of n similar spheres assembled in a cloud is considered, the resistance of each of them will be diminished by an expression proceeding after powers of a the first term of which will be of the order of magnitude $\mu c a^2 \sum \frac{1}{R}$

Thus Stokes law of resistance will not be even true approximately, and the development will cease to be convergent, in general unless $\frac{na}{\delta}$ is small, where δ denotes a kind of mean distance magnitude comparable with the linear dimensions of the cloud of spheres.

The same result follows by the following simple reasoning. Imagine a spherical cloud of radius δ, containing n spherical particles each of radius a and density σ, suspended in a medium of viscosity μ, of negligible density, say a cloud of minute drops of water in air, for instance. Then currents will take place in the spherical cloud and it will attain a certain velocity as a whole, which may be calculated approximately after the general formula (   ) just as if the cloud would form a homogeneous medium of density $\frac{na^3}{\delta^3}$ and of the same viscosity μ as the outer medium, namely

$\frac{4}{15} \cdot \frac{3}{9} \frac{\sigma}{5} \frac{n a^3 g}{\delta \mu}$; this mass movement is superposed upon the displacement of the particles relative to the moving medium. (the downward velocity will be much increased and) Thus evidently (Stokes law cannot be true even approximately) unless $\frac{na}{\delta}$ is a small in comparison to unity taking place with velocity of $\frac{2}{9} \frac{\sigma a^2 g}{\mu}$) This condition shows that Stokes law can be applied only to particles constituting clouds of exceedingly rare distribution transparent consistency. It is easily seen that it would be quite erroneous to apply it in the case of actual fogs, scarce crowding with diminished transparency [as in this case the aggregate cross section of the particles $na^2\pi$ is comparable with the cross section of the cloud $\delta^2\pi$]. (As illustration How cautious we must be in this respect I may mention that the ratio $\frac{na}{\delta}$ amounts to 10 and even 100 for a cubic centimeter cloud produced by JJ Thomson and HA Wilson in their experiments on determination of the ionic charge.

Of course what has been said above applies only to clouds moving in an otherwise unlimited medium. The conditions of motion for a cloud contained in a closed vessel are quite different.

Mr. E. Cunningham has attempted to evaluate the order of magnitude of the correction (Stokes' law in) to be applied to this case. His calculation is founded on the supposition that each particle moves approximately in such a way as if it were contained in a rigid spherical envelope of radius comparable with half the distance to its next neighbours. Now this supposition does not seem quite evident, although we shall see that it leads to a result of the right order of magnitude.

But for We can calculate the resulting motion in quite an exact way, if we consider an homogeneous assemblage of equal spherical particles moving all of them with the same velocity c in the direction of negative X toward an infinite rigid wall which we assume to be the plane YZ.

~~The contribution~~ In order to evaluate β we must know how the particles are arranged. ~~This can be done~~

If we suppose for instance arrangement in ~~cubic~~ rectangular order, we can ~~se~~ get ~~the~~ (easily) value by explicit calculation, ~~for a cube~~ ~~of length~~ by ~~M~~ ~~using the formula~~ integrating over a cube of ~~length~~ height H ~~equal to multiple of R~~ (an approximate) ~~the~~ constructed around the point ξ

$$\iiint \frac{1}{r}\left(1+\frac{x^2}{r^2}\right) dx\,dy\,dz = 8H^2\left[\log(1+\sqrt{3})-\tfrac{1}{2}\log 2-\frac{\pi}{12}\right]$$

It is sufficient to take H equal to a small multiple of R as the expression for β is rapidly converging with the extension of the limits of integration. In this way I have found the approximate value $\beta = 3{\cdot}09$

whence the resistance ~~of the~~ (upon one) particle will be

$$6\pi\mu a c\left[1+\frac{3}{4}\frac{a\beta}{R}\right] = 6\pi\mu a c\left[1+2{\cdot}32\,\frac{a}{R}\right]$$

Prof. Cunningham ~~result~~ (estimates) for an equilateral arrangement ~~is $6\pi\mu a c\left[1+3{\cdot}67\frac{a}{R}\right]$ which is of the same~~ ~~and~~ concluded by his reasoning that the value of the ~~factor~~ (expression) ~~this~~ must be included between the limits ~~factor~~ ~~$3{\cdot}67\frac{a}{R}$ and~~ coefficient of $\frac{a}{R}$ $3{\cdot}67$ and $4{\cdot}5$

(2) order of magnitude.

Our result agrees to the order of magnitude with the we can apply H.A. Lorentz's formula before alluded to and we proceed as follows:

In this case a moving spherical particle of radius a whose coordinates are $x\,y\,z$ is producing at a point ξ situated on the X axis a velocity component

$$u = -\frac{3}{4}\frac{ac}{r}\left[1+\frac{(\xi-x)^2}{r^2}\right] + \frac{3}{4}\frac{ac}{\rho}\left[1+\frac{x^2+\xi^2}{\rho^2}+\frac{6x\xi(x+\xi)^2}{\rho^4}\right]$$

where $r = \sqrt{(x-\xi)^2+y^2+z^2}$ and $\rho = \sqrt{(x+\xi)^2+y^2+z^2}$.

The first part of this expression containing $r=\sqrt{(x-\xi)^2+y^2+z^2}$ is the component of direct motion according to the usual formula of Stokes; the second part is the component caused by reflection at the plane YZ, transformed according to H.A. Lorentz and containing the distance between the point ξ and the reflected source:

$\rho=\sqrt{(x+\xi)^2+y^2+z^2}$

Higher powers of $\frac{a}{r}$ have been neglected, as we confine ourselves to the first approximation.

The total current produced in the point ξ by the motion of all particles is equal to $U=\sum u$ where the sum is to be extended over all their values of $x\,y\,z$. Now we might think it justified to replace the summation by an integration, considering that one particle corresponds to the space R^3, (if R denotes the distance between the particles). In this case the result would be very simple, for we should have

$$U = \frac{1}{R^3}\iiint u\,dx\,dy\,dz$$

and the integrals of the separate terms constituting u can be evaluated explicitly, if we extend them to a cylinder with YZ as basis, of height h and of radius G; then we can use the well known expression for the potential of a disk in points of its axis, and the expressions derivable from it by differentiation with respect to ξ; by these means we find the unexpected result that the integral current U is zero, if we extend the summation to an infinite value of G.

But in reality U is not defined by integration but by summation. Evidently both operations lead to the same result for distant parts of the space but not for those parts whose distance from the point ξ is comparable with the distances R between two particles. Therefore the resultant current U in points at a great distance (in comparison with R) from the wall will be given by:

$$U = +\frac{3}{4}\frac{ac}{R}\beta$$

where $\beta = \left(\sum \frac{R}{r}\left(1+\frac{x^2}{r^2}\right) - \frac{1}{R^3}\iiint \frac{R}{r}\left(1+\frac{x^2}{r^2}\right)dx\,dy\,dz\right)$ [to be extended over a space of dimensions great in comparison with R]

is a numerical factor. This current is directed along the positive X and has the effect therefore to increase the resistance.

explicit numerical calculation for a cube of $5R$

the provisional value: $\beta = 2{\cdot}09$, whence the resistance of the particle will be $6\pi\mu ac\left[1+2{\cdot}32\frac{a}{R}\right]$

which may be compared with Cunningham's value $6\pi\mu ac\left[1+3{\cdot}67\frac{a}{R}\right]$

We see that these developments would be divergent for an infinite number of spheres. It is ~~quite~~ evident that (for instance) an infinite row of ~~similar spheres~~ special particles at equal distances would acquire infinite velocity by virtue of their gravity, as also an infinite cylinder would behave in the same way. A fortiori this applies to two (or three) dimensional infinite assemblages.

F $F = 6\pi\mu a c\left[1 + \frac{3}{4}\frac{\rho}{?}\frac{a}{R}\right]$

28

Our formula (…) would apply of course also if the particles were arranged in a different way but

(1) then the numerical value of ρ (…) will be different.

— Of course it must be emphasized that we have considered only the terms with the first power of $\frac{a}{R}$; in order to evaluate quadratic terms, it would be necessary to consider how the disturbed motion is modified by the presence of any other particle, being at rest, and besides the reaction of the motion of the particle by reflexion at all other particles ought to be taken into account.

The general result of our calculation, it shows that (Stokes' law) is undergoing but comparatively small corrections, if applied to the particles of a uniform cloud filling a closed vessel. But it is important to note that things will go on in quite a different way if the cloud is not of quite uniform density, or if it does not fill the whole empty space between the walls.

Then as a rule convective currents will arise, the velocity of which is superposed on the movement of the particles and which in certain cases may be of preponderent influence. Their velocity may be calculated approximately by considering the medium as a homogeneous liquid subjected to certain forces, the intensity of which (per unit volume) correspond to the aggregate force acting on the particles contained in it.

Consider for instance an electrolyte in an electric field. As it is conducting in accordance with Ohm's law, the average electric density is zero and no currents will take place. But in bad liquid conductors with deviations from Ohm's law convective currents may arise. (They have been studied during long times for instance by Warburg. which may influence materially also the apparent value of the conductivity)

Similar movements may be produced in ionised gases and I think more attention ought to be paid to the possible deviations from Stokes law in this case than usually is done. If the dimensions of the space filled with gas are small and if the ionization is weak, no appreciable effect will be produced of course, but in experiments where the saturation current of strong radioactive material is observed between condenser plates wide apart, these phenomena may be of importance as producing an apparently greater mobility of the ions than under normal circumstances

(*) F. inst. Rutherf. Radioactivity p. 84, p. 35

Stokes' Law?

Cambridge 1912

There is one more application of the theoretical methods explained ~~mentioned~~ before which may be mentioned.

Imagine a two dimensional infinite assemblage of equal spherical particles, arranged (in ~~a plane~~ quadratic order ?) in the plane $x = l$ ~~parallel to~~ (the YZ plane ~~which~~ again may be supposed to be a rigid ~~or a fixed~~ wall.

Now let all these particles be moving in the direction parallel to Y with the velocity c; what motion will be produced in the surrounding liquid, what will be the resistance experienced by every particle? ~~opposing to the movement of~~

~~The method of explicit calculation~~

~~The explicit calculation can be effected by means of the formula ... defining after~~

According to Lorentz the motion produced by a sphere of radius a moving with velocity c parallel to a fixed wall is, with neglection of higher powers of $\frac{a}{l}$ which we suppose a small quantity:

$$v = \frac{3}{4} Rc \frac{1}{r}\left[1 + \frac{y^2}{r^2}\right] - \frac{3}{4} Rc \frac{1}{r_1}\left[1 + \frac{y^2}{r_1^2}\right] - \frac{3}{2} Rc \frac{x(x+\xi)}{r_1^3} + \frac{9}{2} \frac{Rc x y^2 (x+\xi)}{r_1^5}$$

where the first term is the direct current, according to Stokes, while the terms containing r_1 represent the current reflected by the wall, just as in the former example before.

We might also in this case calculate the resultant current in the point ξ by forming Σv over all values of x and derive therefrom the resistance of a single particle. But we shall content ourselves with the following remarks.

The resultant motion must depend on the ratio of the radius of each particle to their mean distance.

In the extreme case where the particles are so crowded as to touch one another, a lamellar flow will take place in the liquid between the fixed wall and the plane $x = l$ with velocity $v = \frac{cx}{l}$, while on the other side of the plane $x = l$ the liquid will be dragged along by the moving plane particle with the constant velocity c.

The frictional force per unit of surface of the plane $x = l$ is evidently equal to $\frac{\mu c}{l}$; therefore the resistance experienced by each particle is

$$F = \frac{\mu c R^2}{l}$$

it is much smaller of course than the Stokes law would indicate instead of the Stokes resistance $6\pi\mu c a$ which is as R is of the order of a, but l is supposed to be of higher order.

Now consider the other extreme case where the distances R between the particles are so great that Stokes' law is approximately valid, the condition will be that R be of order l.

So we have $F = 6\pi\mu c a$

On the other side let us calculate the resultant motion (in the liquid) for a point at infinite ~~great~~ distance from the wall $\lim \xi = \infty$

For such points the summation mentioned above can be replaced by integration, besides we compute ~~the differences~~

$\frac{1}{r} - \frac{1}{r_1}$ ~~can be developped put equal to~~ $= \frac{2l\xi}{r^3}$; $\frac{1}{r^3} - \frac{1}{r_1^3} = \frac{6l\xi}{r^5}$, thus the above expression simplifies into:

$$V_\infty = \Sigma v = \frac{3}{4} Rc \iint \left\{\frac{2l}{r^3} + \frac{6ly^2}{r^5}\right\} - \frac{3}{2} Rc \frac{l\xi}{r^3} + \frac{9}{2} \frac{Rcly^2\xi}{r^5}$$

$$= 9 \frac{acl\xi}{R^2} \iint \frac{y^2\, dy\, dz}{\sqrt{\xi^2 + y^2 + z^2}^{\,5}}$$

The integral can be transformed by putting $y = \rho \sin\varphi$, $z = \rho\cos\varphi$, $dy\,dz = \rho\, d\rho\, d\varphi$

into $\int_0^\infty \int_0^{2\pi} \frac{\rho^3 \sin^2\varphi\, d\varphi}{\sqrt{\xi^2 + \rho^2}^{\,5}} = \pi \int_0^\infty \frac{\rho^3 d\rho}{\sqrt{\xi^2+\rho^2}^{\,5}} = \frac{2\pi}{3\xi}$ and we get $V_\infty = \frac{6al\pi c}{R^2}$

~~Of Cunningham assumes for arrangement in~~ [an equilibrial] | However, The practical application of this formula is rather questionable ~~however~~ for

(3) It is interesting (to note) that the average value of β for a particle whose position ~~of spherical~~ ~~is~~ ~~position~~ ~~accidental~~ relatively to the other particles is ~~chosen~~ defined by accident would be zero; thus it follows that to this order of approximation Stokes' law would apply (for the particles of a cloud) on an average without any correction.

That seems quite natural as the average current U (of liquid) in the crossection must be zero and

By comparing this expression with F we get

$$\frac{V}{\infty} = \frac{F}{R^2}\frac{l}{\mu}$$

that means that the liquid at a great distance from the wall will be dragged along in a parallel direction to it with such a velocity as if the force corresponding to unit surface $\frac{F}{R^2}$ would act not on a crowd of particles but on a plane distant by l from the fixed wall. Now if we suppose a greater number of similar layers we see that generally under condition of approximation — Without going into further details, I may only mention that this result has an important bearing on the theory of electric endosmose which I am going to explain elsewhere with the full details.

I may conclude with some remarks about the influence of the inertia terms in the hydrodynamical equations, which are neglected as well in Stokes original formula as in the preceding reasonings. Oseen still
It is well known that this neglection is justified only if under condition that the ratio $\frac{a v 6}{\mu}$ is small in comparison to unity. The recent experiments of Dr Arnold which have shown that Stokes' law is valid for spheres moving under influence of gravity, provided their radius is smaller than 0.6 $\bar{r}$, where the critical radius $\bar{r}$ is defined by $\frac{\bar{r} v 6}{\mu} = 1$; this means as much that the ratio $\frac{a v 6}{\mu}$ must be smaller than $(0{\cdot}6)^3 = 0{\cdot}22$.

But it has been proved in an interesting paper by Oseen that the hydrodynamical solution given by Stokes is defective even if this criterion is fulfilled; for at distances r where $\frac{r v 6}{\mu}$ is large, the inertia terms must be of preponderant influence. Oseen himself has given a solution, different from Stokes' equations, which gives better approximation for the currents in those distant parts of the space. However, the resistance of a sphere in a liquid of infinite extent depends only on the state of movement in its immediate neighbourhood, therefore the resistance law of Stokes is not impaired by these results, provided the criterion mentioned above is fulfilled.
The conditions of its validity may be defined more exactly by means of

The inertia terms are of greater importance, as modifying the law of resistance in the case before alluded to where the motion of a greater number of similar spheres is considered. For it is legitimate to use the use of Stokes equations for slow motion only if the forces of reaction between such spheres can be calculated by if they are lying within the space where viscosity is predominant over inertia. Mr. Oseen has recently generalised the calculation of reciprocal interaction of two spheres, given by me, by introducing in it his solution of Stokes' problem. The forces exerted on the two spheres come out unequal in this case and are given by much more complicated expressions. They become identical with the first approximation given by me if the distance R between the spheres satisfies the criterion that $\frac{R v 6}{2\mu}$ is small.
Mr. Oseen thinks this to be a great restriction on the validity of these formulas for experimental purposes, but he omits inadvertently the factor 6 in the above expression. We satisfy ourselves easily that for instance

serious difficulties with experiments of this kind may arise on account of solid surface films as Dr. Arnold has shown so easily produced at liquid surfaces.

Millikan Phys. Review 32 (1911) p. 349
Mr. Keehan p. 381

Millikan Ph. Z. 11, 1097 (1910)

Mc K. Ph. Z. 12, 707 (1911)

M. Knudsen & S. Weber Ann. d. Ph. 36, 981, 1911

Pr. R. S. 83, 357 (1910)

known how the ultimate interaction between gas molecules and the surface of the sphere takes place. If the molecules rebound like elastic spheres we should get (in accordance with Cunningham):

$$F = \frac{4}{3}\sqrt{\frac{8}{3\pi}}\, a^2 \pi \rho c V$$

where c is the square root of the mean square of molecular velocity. The empirical numerical coefficient, as following from Mr. Keehan's, Knudsen's experiments is considerably larger, it amounts to 1·65 (according to Knudsen and S. Weber) or 1·84 according to Mr. Keehan instead of 1·23 as would follow for elastic impacts. Mr. Keehan concludes that molecules are reflected from the surface of the sphere only in normal direction; I think however that his theoretical formula requires a little correction and that these experiments are quite in accordance with the view supported by other researches of this kind by Warburg and Knudsen that a solid surface acts in scattering the impinging molecules irregularly in all directions.

See also M. Reinganum Verh. D. Ph. G. 12, 1025 (1910)

We shall not go into these questions here, however, as they belong to the kinetic theory of gases.

ii

R is

for the

~~is experi~~
(Abstract)
for ~~[illegible]~~
elabora
classi
Osen
F

~~th~~

to D

on th

retio ... two.

Ordinary hydraulic experiment

Stokes law or any of those modifi: cati

31

On the Practical Applicability of Stokes' Law of Resistance, and the Modifications of it Required in Certain Cases.

by

M. S. Smoluchowski, Ph.D., LL.D. Professor of Physics at the University of Lemberg.

---

§1). Stokes' ~~well known~~ law for the resistance of a sphere in a viscous liquid rests, as is well known, on the fundamental assumptions:

I. Slowness of motion, so that the inertia terms in the hydrodynamical equations may be neglected, in comparison with the effects of viscosity,

II. Complete adhesion, without slip, of the liquid to the sphere, this being considered as a rigid body,

III. Unboundedness of the liquid and immobility at infinity.

In the following ~~the question will be considered – some contributions~~ I should like to contribute some remarks on this law ~~and to give~~ ~~will be added to the discussion of the modifications of this law~~ ~~and~~ some modifications, applicable with regard to certain cases of practical importance, where the ~~[illegible]~~ underlying ~~suppositions~~ conditions are ~~to be~~ changed to some extent, which may be of some interest to those who are engaged with research work on subjects connected with the law.

First ~~I~~ let us ~~consider~~ touch briefly the question of slipping, implied in the second of the above assumptions. It is easy to generalise Stokes' calculation, by allowing the liquid to slip along

case to the fourth order of approximation (including terms with $(\frac{R}{a})^4$).[1)]

In a somewhat similar way Ladenburg[2)] calculated the ~~amount of~~ resistance experienced by a sphere, when moving along the axis of an unlimited cylindrical tube, and his result, indicating an increase in comparison with the usual formula of Stokes in the proportion of $1 : 1 + 2{\cdot}4 \frac{R}{\rho}$, (where ρ = radius of tube), has been verified with very satisfactory approximation by his own experiments and by those of Mr. Arnold.

2) R. Ladenburg, Ann. d. Phys. 23, 447 (1907).

§5) [In a paper published last year[3)] I have pointed to some interesting results concerning the motion of a greater number of similar spheres and I may be allowed to extend these investigations a little further now]. Imagine a sphere of radius R moving with the velocity c ~~in the direction~~ of along the X axis, its centre being situated at the distance x from the origin. It would produce at the point P (with coordinates ξ, η, ζ) certain current velocities $u_0\ v_0\ w_0$, of order $\frac{Rc}{r}$, defined by Stokes' equations, if the fluid be unlimited. But if we assume this point P to be the centre of a solid sphere of radius $\mathcal{B}$, we have to superpose a fluid motion u_1, v_1, w_1 chosen so as to annull the velocities of the primary motion at the points of this sphere and satisfying the condition of rest for infinity.

3) M. Smoluchowski, Bull. Acad. Scienc. Cracovie 1911 p. 28.

1) J. Stock, Bull. Acad. Scienc. Cracovie 1911 p. 18. In Millikan's determinations of the ionic charge this increase of resistance, arising from the presence of the condenser plates, may produce an increase of the order of one thousandth.

Now let us apply this method to the case ~~W~~ where a greater number of similar

~~an investigation~~

spheres is in motion, ~~a subject which I have treated~~ (in a paper publ. —

a little further now

and extend in this way (an investigation which I had begun :)

$p_u = \mu \frac{\partial u}{\partial y}$

$\beta = \frac{\mu}{\gamma} \quad \| \mu = \varepsilon\lambda$

Lamb:

$F = 6\pi\mu c R \dfrac{1 + 4\left(\frac{\mu}{\beta R}\right) + 6\left(\frac{\mu}{\beta R}\right)^2}{\left(1 + \frac{3\mu}{\beta R}\right)^2}$

$F = \underbrace{\frac{4}{3}\sqrt{\frac{8}{3\pi}}}_{=1.23} R^2 \pi \rho c V$ elastic $= \dfrac{6\pi\mu c}{1 + 1.5\,\frac{\lambda}{a}}$

$= \dfrac{6\pi\mu c}{1 + 1.2\,\frac{\lambda}{a}}$ Mc Keehan (all directions) $= \frac{5}{3}\sqrt{\frac{8}{3\pi}} \dots = 1.535$

$= \dfrac{6\pi\mu c}{1 + 1.05\,\frac{\lambda}{a}}$ " (normal) $= \quad = 1.76$

$= \dfrac{6\pi\mu c}{1 + 1.00\,\frac{\lambda}{a}}$ empiri. $= 1.84$

Smol. : all directions (ang. unchanged) $= R^2 \pi \rho c V. \; \frac{13}{9}\frac{4}{3}\sqrt{\frac{8}{3\pi}} = 1.78$

all directions (ang. changed) $= \frac{16}{9}\sqrt{\frac{8}{3\pi}} \dots = 1.64$

empirical Knudsen $= 1.65$

$u = \frac{3}{4} c \left\{ \frac{R}{r} + \frac{R x^2}{r^3} \right\} + \frac{1}{4} c \left(\frac{R^3}{r^3} - \frac{R^3 x^2}{r^5} \right)$

$v = \frac{3}{4} c \left\{ \frac{R}{r^3} xy - \frac{R^3 xy}{r^5} \right\}$

$w = \frac{3}{4} c \left\{ \frac{R xz}{r^3} - \frac{R^3 xz}{r^5} \right\}$

Eq. (7) $u = \frac{3}{4} c \frac{R}{r} \left[1 + \left(\frac{\xi - x}{r} \right)^2 \right]$

$v = \frac{3}{4} c \frac{R(\xi - x)\eta}{r^3}$

$w = \frac{3}{4} c \frac{R(\xi - x)\zeta}{r^3}$

Langevin Comptes Rendus 146 p 530 (1908)

$$m \frac{d^2x}{dt^2} = -6\pi\mu a \frac{dx}{dt} + X$$ = force complémentaire qui maintient l'agitation ~~le mouvement~~ que sans elle la résistance visqueuse finirait par arrêter

~~$x\frac{d^2x}{dt^2}$~~ multipliée par x

$$\frac{m}{2}\frac{d^2(x^2)}{dt^2} - m\left(\frac{dx}{dt}\right)^2 = -3\pi\mu a \frac{d(x^2)}{dt} + X_x$$

$$\frac{d(x^2)}{dt} = 2x\frac{dx}{dt}$$

$$\frac{d^2(x^2)}{dt^2} = 2\left(\frac{dx}{dt}\right)^2 + 2x\frac{d^2x}{dt^2}$$

grand nombre de partic. identiques et prenons la moyenne des équat(4)

la moyenne des termes X_x est évidemment nulle, à cause de l'irrégularité des actions complémentaires X, et il vient, en posant $z = \frac{d(x^2)}{dt}$

$$n\frac{mc^2}{3} = P = R\theta\rho$$

$$\frac{mc^2}{3} = R\theta\frac{\rho}{n} = \frac{R\theta}{N}$$

$$\frac{m}{2}\frac{dz}{dt} + 3\pi\mu a z = \frac{RT}{N}$$

~~et~~

$$z = \frac{RT}{N}\frac{1}{3\pi\mu a} + Ce^{-\frac{6\pi\mu a}{m}t}$$

régime permanent au bout d'un temps de l'ordre $\frac{m}{6\pi\mu a}$ $(= 10^{-8})$

$$\therefore \frac{d(x^2)}{dt} = \frac{RT}{N}\frac{1}{3\pi\mu a}$$

d'où, pour un intervalle de temps τ

$$\therefore \overline{x}^2 - \overline{x_0}^2 = \frac{RT}{N}\frac{1}{3\pi\mu a}\tau$$

le déplacement Δx d'une particule est donné par:

$$x = x_0 + \Delta x$$

et comme ces déplacements sont indifféremment posit. et négat.

$$\overline{\Delta x^2} = \overline{x^2} - \overline{x_0^2} = \frac{RT}{N}\frac{1}{3\pi\mu a}\tau$$

$\not{E}$ mesures de Svedberg approche plus sim. mais incertitude sur le diamètre réel

préférables granules microscopiques de dimensions plus faciles à connaître et pour lesquels l'applic. de la formule de Stokes qui néglige les effets d'inertie du liquide est plus légitime

$$\Lambda = c\sqrt{\frac{2m}{f}} = c\sqrt{\frac{2m}{6\pi\mu R}} = \frac{c\sqrt{m}}{\sqrt{3}\sqrt{\pi\mu R}}$$

$$\frac{8}{3\sqrt{3}} = \sqrt{\frac{64}{27}}$$

$$v e^{-v} + 2\frac{v^2 e^{-v}}{2!} + 3\frac{v^3 e^{-v}}{3!} + \cdots \quad - v =$$

$$v e^{-v}\left[1 + \frac{v}{1!} + \frac{v^2}{2!} + \cdots\right]$$

$$\underbrace{\qquad\qquad}_{e^{+v}}$$

$$(0-v)^2 e^{-v} + (1-v)^2 v e^{-v} + (2-v)^2 \frac{v^2 e^{-v}}{2!} + \cdots$$

$$= \sout{0 \cdot e^{-v} + v e^{-v} + v^2}$$

$$e^{-v}\left\{0^2 + 1^2 v + \frac{2^2}{2!}v^2 + \frac{3^2}{3!}v^3 + \frac{4^2}{4!}v^4 + \cdots\right\}$$

$$- 2v\left(0 + v + \frac{2v^2}{2!} + \frac{3v^2}{3!} + \cdots\right\}$$

$$+ v^2\left(0 + v + \frac{v^2}{2!} + v^3 \cdots\right\}$$

$$f(v) \quad 0^2 + v + \frac{2v^2}{1!} + \frac{3v^3}{2!} + \frac{4v^4}{3!} + \cdots$$

$$\int \left(\frac{f(v)}{v}\right) dv = \quad \frac{v^3}{2!} + \frac{v^4}{3!} + \quad = e$$

T Uran = 290.10⁷

$$\frac{m}{2}\frac{d(x^2)}{dt}\Big|_0^T - \int_0^T m\left(\frac{dx}{dt}\right)^2 dt = -3\pi\mu a\, x^2\Big|_0^T + \int_0^T X_x\, dt$$

13	$\frac{1}{2}$	5·6 · 1·21
		1 1 8
3	$\frac{1}{2}$	6 8
5·6		0·24
		1·21

Lamb

Wien

Kuenig

$$\Sigma(x)^2 - \Sigma x_0^2$$

$$\Sigma(x_0 + \Delta x)^2 - \Sigma x_0^2$$

$$= ~~2\Sigma x_0 \Delta x~~ + \Sigma \Delta x^2$$

$$e^{-\frac{N}{H\theta}A} \qquad A = a\varepsilon^2$$

$$dW = \text{Konst.}\; e^{-\frac{N}{H\theta}a\varepsilon^2}\, d\varepsilon$$

$$\overline{\varepsilon^2} = \frac{\int \varepsilon^2 e^{-\beta\varepsilon^2} d\varepsilon}{\int e^{-\beta\varepsilon^2} d\varepsilon} = \frac{1}{2\beta}$$

$$a\overline{\varepsilon^2} = \frac{1}{2}\frac{H\theta}{N}$$

$$= \frac{1}{3}\frac{mc^2}{2}$$

$$= 10^{-16}\theta.$$

$$m\frac{nc^2}{3} = R\theta$$

$$\frac{mc^2}{3} = \frac{H\theta}{N}$$

$$N = 6.10^{23}$$

~~$H = 8.10$~~

$$8{,}3.28$$
$$166$$
$$664$$
$$\underline{232}$$

$$H = 2{,}3.10^9$$

$$R = 8{,}3.10^7$$ (~~8.20~~)

$$H = \frac{10^6 . 2}{\text{~~0.0012~~}.\, 270}$$

$$0{,}00009$$
$$0{,}00243$$

$$= \frac{10^9}{0{,}0012} = 8.10^9$$

$$\frac{2\,8.10^9.\,\text{~~273~~}\,188}{2\;8.10^{23}\;3} = 18.10$$

$$\frac{1}{2}\frac{2{,}3.10^9}{6.10^{23}}$$

Kurve bezogen

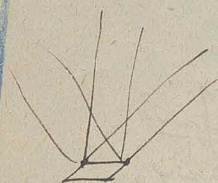
$T = 290.10^9$

całość   momentu w razie zderzenia

$$\frac{4\pi v_0^2 e^{-\frac{v_0^2}{\alpha^2}} dv_0}{\alpha^3\sqrt{\pi}} \quad \frac{\sin\varphi\, d\varphi}{2} \quad v \quad \underbrace{v\cos\psi}_{(v_0\cos\varphi + U)}$$

$$v^2 = U^2 + v_0^2 + 2Uv_0\cos\varphi$$

$$\frac{1}{2}\int (v_0 + U\cos\varphi)(v_0\cos\varphi + U)\sin\varphi\, d\varphi$$



Tak jak gdyby w ściance okienko i gaz z drugiej strony wylatywał; przeciętny moment w kierunku normalnym:

$$\frac{\int \frac{\sin\varphi\, d\varphi}{2}\, d\sigma\cos\varphi \cdot v \cdot n \cdot v\cos\varphi}{\int \frac{\sin\varphi\, d\varphi}{2}\, d\sigma\cos\varphi \cdot v \cdot n} = v\,\frac{\int_0^{\frac{\pi}{2}} \sin\varphi\cos^2\varphi\, d\varphi}{\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi} = \frac{2v}{3}$$

zatem przeciętny moment w kierunku v:

$$\frac{\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \cdot \cos\theta\, \frac{2v}{3}}{\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta} = \frac{4}{9}v$$

tożsamy Akkommod. Coeff. dla energii $= 0$
dla kierunku ruchu $= 1$

zatem przychodzi $(v + \frac{4}{9}v)\cos\varphi$ więc wszystko powiększone w stosunku $\frac{13}{9}$

$$\frac{13}{9}\,\frac{4}{3}\sqrt{\frac{8}{3\pi}}$$

1·23 . 13
369
1599 : 9 =
69 69
1·781

$$1{\cdot}5 \cdot \frac{9}{13} = 13{\cdot}5 : 13 = 1{\cdot}00 \quad \frac{\lambda}{2}$$

---

w razie zaś jeśli także energia się wyrównuje całkowicie i jeśli odbite cząsteczki posiadają prędkości według prawa Maxwella rozłożone bez względu na prędkość wpadających

$$\frac{\int 4\pi v^3 e^{-\frac{v^2}{\alpha^2}} dv}{\int 4\pi v^2 e^{-\frac{v^2}{\alpha^2}} dv} = \frac{2\alpha}{\sqrt{\pi}}$$

w takim razie przychodzi

$$\left(v + \frac{4}{9}\frac{2\alpha}{\sqrt{\pi}}\right)\cos\varphi$$

Akk. dla energii i także dla kierunku ruchu $= 1$

$$\int_0^{\pi} \frac{\sin\varphi}{2} d\varphi\, (v_0 + U\cos\varphi)(v_0\cos\varphi + U) + \frac{4}{9}\frac{2\alpha}{\sqrt{\pi}} \underbrace{\int_0^{\pi} \frac{\sin\varphi\, d\varphi}{2} (v_0\cos\varphi + U)}_{U}$$

$$\left[v_0^2\cos\varphi + Uv_0(1+\cos^2\varphi)\right]$$

$$Uv_0\left(1 + \frac{1}{3}\right)$$

wtedy zatem $1{\cdot}5 \cdot \frac{3}{4} = 1{\cdot}125\ \frac{\lambda}{a}$

$= 1{\cdot}64$

$$\frac{4}{3}Uv_0 + \frac{8}{9\sqrt{\pi}}\alpha U$$

$$U\left[\frac{4}{3}\,\frac{\int v^3}{\int v^2} + \frac{8\alpha}{9\sqrt{\pi}}\right] = U\left(\frac{8}{3} + \frac{8}{9}\right)\frac{\alpha}{\sqrt{\pi}} = U\,\frac{8}{3}\left(1 + \frac{1}{3}\right)\frac{\alpha}{\sqrt{\pi}}$$

$$\frac{\int v^3}{\int v^2} = \frac{2\alpha}{\sqrt{\pi}}$$

$$= U_c\,\frac{32}{9}\sqrt{\frac{2}{3\pi}}$$

$$= U_c\,\frac{16}{9}\sqrt{\frac{8}{3\pi}}$$

$$\frac{1}{r}-\frac{1}{\rho}=\frac{2a^2}{r^3}$$

$$\sum \frac{1}{r^3} = y^{-2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos d\varphi = \frac{2}{y^2}$$

$$2\,\frac{2}{\delta^3}\Big[\underbrace{1+\frac{1}{4}+\frac{1}{9}\cdots}_{1.65} = \frac{6.6}{\delta^3}$$

$$= \frac{(2a)^2}{\delta^3}\cdot 3.3$$

$$= \frac{13.2\,a^2}{\delta^3}$$

Sdýy cetkem gesto:

$$4\pi\sigma = \frac{\varphi_1-\varphi_0}{2a} = \frac{2(\varphi_1-0)}{2a} = \frac{\varphi_1}{a}$$

$$\varphi_1 = 4\pi\sigma a$$

$$= \frac{4\pi a}{\delta^2} \quad \text{o ile} \ldots \quad \delta \ll a$$

$$S=\sum_{x=1}^{\infty}\frac{1}{x^2}$$

$$1+z+z^2+z\cdots = \frac{1}{1-z} \qquad \frac{[1-\cos\varphi+i\sin\varphi]}{[1-\cos\varphi]^2+\sin^2\varphi} =$$

$$1+\cos\varphi+\cos 2\varphi+\cdots = \frac{1-\cos\varphi}{2(1-\cos\varphi)} = \frac{1}{2}$$

$$+i[\sin\varphi+\sin 2\varphi+\cdots] = i\frac{\sin\varphi}{2(1-\cos\varphi)} = \frac{i}{2}\operatorname{ctg}\frac{\varphi}{2}$$

$$\int_0 dx\,[\cos\alpha x+\sin 2\alpha x+\cdots] = \frac{1}{2}\int \operatorname{ctg}\frac{\alpha x}{2}\,dx$$

$$1+e^{-z}+e^{-2z}+\cdots = \frac{1}{1-e^{-z}}$$

$$\int_0^{\infty} dx\; e^{-\alpha x}+e^{-2\alpha x}\cdots = \frac{1}{1-e^{-\alpha x}}-1$$

$$\frac{1}{\alpha}+\frac{1}{2\alpha}+\frac{1}{3\alpha}\cdots = \int \frac{e^{-\alpha x}}{1-e^{-\alpha x}}\,dx$$

$$1+\frac{1}{2}$$

Jb Nr 175 ~~[illegible]~~ Wie ein Buch entsteht, Unger

138 D. gewerbl. Rechtsschutz i. Deutschl. Tolksdorf

30 Meeresforschung u. Meeresleben Janson

255 Nautik Möller

216 Die Uhr Bock.

moment $= -\theta\varphi$

Potencjał $= -\theta \frac{\varphi^2}{2}$

$\sqrt{\bar{\varphi}^2} = \sqrt{\frac{RT}{N\theta}}$

$\varphi = 10^{-4} = 10^{-4}.60.60.60 =$ ~~[illegible]~~ $= 20''$

$=$ ~~[illegible]~~ 1 mm na podziałce odległej 0.5 m

~~[illegible]~~

$\theta = 5.10^{-6}$ (CGS)

$= \frac{\pi r^4 E}{4(1+\mu)l} \neq \frac{r^4 E}{2l}$

biorąc $l = 100$

$10^{-3} = r^4 E$

$E =$ ~~2.10^{12} dla platyny~~

$= 0{,}5.10^{12}$ (kwarc)

$r^4 = \frac{10^{-3}}{0{,}5.10^{12}} = 2.10^{-15} = 20.10^{-16}$

$r = < 2.10^{-4} =$ ~~[illegible]~~ 0,002 mm

wytrzymałość na zerwanie $\pi r^2.\ 9.\ 10^9 = 30.\ 4.10^{-8}.10^9$

$= 1200$ dyn

$= 1$ gram

biorąc $r = 2.10^{-4}$ mm

otrzymałoby się wytrzymałość $=$ 0,01 gram

a wychylenia 60°