CONTINUATIO CIRCULI QUADRATURÆ NOVISSIMÆ ET BREVISSIMÆ.

PROBLEMA IV.

§. 13. *Determinare partes constituentes summam, cujus numerator est aggregatum ex utroque factore denominatoris.*

Sit denominator summæ $= mo$ & numerator $= o + m$, h. e. sit ille aggregatum ex utroque factore o & m denominatoris mo : erit summa ipsa $= \frac{o+m}{mo} = \frac{o}{mo} + \frac{m}{mo} = \frac{1}{m} + \frac{1}{o}$

Theorema : *Si numerator summæ est aggregatum ex utroque factore denominatoris ; necesse est, ut unus factor sit denominator unius partis, alter factor denominator alterius, & unitas numerator utriusque partis.*

§. 14. *Corollarium.* Si itaque summa fuerit $= \frac{d+a}{ad}$; erunt partes constituentes excessum & defectum $\frac{1}{a}$ & $\frac{1}{d}$, & quidem (per § 4.) prior excessus & posterior defectus.

§. 15. Scholion. *Hanc veritatem esse indubitatam, colligi potest ex investigatione lunulæ respondentis diam. quadrato* $= 1$, *quam suppono nondum esse inventam. Sit itaque ratio hujus quadrati ad lunulam excessiva, ut* $4 : 1\frac{1}{2} = 8 : 3$ *& defectiva ut* $4 : \frac{2}{3} = 12 : 2$; *erunt lunulæ* $\frac{3}{8}$ *&* $\frac{2}{12} = \frac{36}{96}$ *&* $\frac{16}{96}$: *consequenter* $\frac{35}{96} - \frac{16}{96} = \frac{20}{96}$ *summa excessus & defectus* (§ 2.), *cujus numerator est* $=$ d+a $= 12+8$. *Ergo* (*per* § 4.) *excessus est* $\frac{1}{8}$ *& defectus* $\frac{1}{12}$. *Sit deinde ratio excessiva, ut* $4 : 1\frac{1}{6} = 24 : 7$ *& defectiva, ut* $4 : \frac{6}{7} = 28 : 6$: *erunt lunulæ* $\frac{7}{24}$ *&* $\frac{6}{28} = \frac{196}{672}$ *&* $\frac{144}{672}$: *consequenter* $\frac{196}{672} - \frac{144}{672} = \frac{52}{672}$ *summa excessus & defectus, cujus numerator est* $=$ d+a $= 28 + 24$. *Ergo excessus est* $\frac{1}{24}$ *& defectus* $\frac{1}{28}$. *Jam cum hi excessus subducti ex lunulis excessivis, & defectus additi ad defectivas, manifestent utrobique lunulam* $= \frac{1}{4}$: *consequenter rationem ejus ad diam. quadratum eandem, quæ reperitur per Theorema* Hippocratis ; *dubitari nequit, quin excessus & defectus per Problema III. legitime determinentur. Pergamus jam à cognitis ad incognita: Sit ratio quadrati diametri* $= 1$ *ad segmentum ei respondens excessiva, ut* $64 : 9\frac{1}{8} = 512 : 73$

&

& *defectiva, ut* $64 : 8\frac{3}{4} = 256 : 35$: *erunt segmenta* $\frac{73}{512}$ & $\frac{35}{256} = \frac{18688}{131072}$ & $\frac{17920}{131072}$, *consequenter* $\frac{18688}{131072} - \frac{17920}{131072} = \frac{768}{131072}$ *summa excessus & defectus, cujus numerator est* $= d+a = 256 + 512$. *Ergo excessus est* $\frac{1}{512}$ & *defectus* $\frac{1}{256}$: *consequenter segmentum verum* $\frac{72}{512} = \frac{9}{64}$ *vel* $\frac{36}{256} = \frac{9}{64}$ & *ad quadratum diam. ut* $\frac{9}{64} : 1 = 9 : 64$. *Sit deinde ratio excessiva, ut* $64 : 9\frac{1}{4} = 256 : 37$ & *defectiva, ut* $64 : 8\frac{2}{3} = 192 : 26$: *erunt segmenta* $\frac{37}{256}$ & $\frac{26}{192} = \frac{7104}{49152}$ & $\frac{5655}{49152}$: *consequenter* $\frac{7104}{49152} - \frac{6656}{49152} = \frac{448}{49152}$ *summa excessus & defectus, cujus numerator est* $= d+a = 192+256$. *Ergo excessus est* $\frac{1}{256}$ & *defectus* $\frac{1}{192}$: *consequenter segmentum verum* $\frac{36}{256} = \frac{9}{64}$ *vel* $\frac{27}{192} = \frac{9}{64}$ & *ad quadratum diametri, ut* $\frac{9}{64} : 1 = 9 : 64$. *Est igitur diameter ad periph. ut* $8 : 25$ (§ 8.).

PROBLEMA V.

§ 16. *Determinare tam excessum, quam defectum summæ, cujus numerator est præcise aggregatum ex multiplis denominatorum* $d+a$; *vel ex multiplo denominatoris unius & simplo alterius.*

Sit coefficiens denominatoris d, h. e. numerus indicans, quoties ille fuit sibimet additus, $= m$, & coefficiens denominatoris $a = l$: erit summa excessus & defectus $= \frac{md+la}{ad} = \frac{md}{ad} + \frac{la}{ad} = \frac{m}{a} + \frac{l}{d}$. Sit deinde numerator summæ $= md+a$; vel $d+la$, h.e. sit aggregatum ex multiplo denominatoris d ac simplo denominatoris a, & vice versa: erit summa ipsa in 1mo casu $= \frac{md+a}{ad} = \frac{md}{ad} + \frac{a}{ad} = \frac{m}{a} + \frac{1}{d}$, & in secundo $= \frac{d+la}{ad} = \frac{d}{ad} + \frac{la}{ad} = \frac{1}{a} + \frac{l}{d}$; ex quo fluit

Regula: *Si numerator summæ est aggregatum ex multiplis denominatorum* d+a *ita, ut non possit resolvi in plura, quam in 2 multipla; vel si ille est aggregatum ex multiplo denominatoris* d *ac simplo denominatoris* a, *& vice versa; tunc coefficienti denominatoris* d, *vel si nullus adest, unitati subscribatur denominator quantitatis excessivæ & habebitur excessus: contra coefficienti denominatoris* a, *vel si nullus adest, unitati subscribatur denominator quantitatis defectivæ, & habebitur defectus.*

§. 17 Scholion I. *Ad illustrandum & confirmandum hoc problema, investigetur denuo lunula respondens diam. quadrato* $= 1$: *Sit itaque ratio hujus quadrati ad lunulam excessiva, ut* $4 : 1\frac{2}{9} = 36 : 11$, & *defectiva ut* $4 : \frac{5}{8} = 32 : 5$: *erunt lunulæ per utramque rationem inventæ* $\frac{11}{36}$ & $\frac{5}{32} = \frac{352}{1152}$ & $\frac{180}{1152}$, *consequenter* $\frac{352}{1152} - \frac{180}{1152} = \frac{172}{1152}$ *summa excessus & defectus, cujus numerator est* $= 2d+3a = 64+108$. *Iam cum coefficiens denominatoris* d *sit* 2, & *denominatoris* a *sit* 3; *scribatur sub* 2 *denominator quantitatis excessivæ, & habebitur excessus* $\frac{2}{36}$, *sub* 3 *autem scribatur denominator quantitatis defectivæ &*

habe-

habebitur defectus $\frac{3}{32}$. *Sit deinde ratio excessiva, ut* $4 : 1\frac{1}{4} = 16 : 5$ *& defectiva, ut* $4 : \frac{1}{5} = 20 : 1$: *erunt lunulæ* $\frac{5}{16}$ *&* $\frac{1}{20} = \frac{100}{320}$ *&* $\frac{16}{320}$, *consequenter* $\frac{100}{320} - \frac{16}{320} = \frac{84}{320}$ *summa excessus & defectus, cujus numerator est* $= d + 4a = 20 + 64$. *Iam cum denominator* d *nullo coefficiente sit affectus; subscribatur unitati denominator quantitatis excessivæ, ut prodeat excessus* $\frac{1}{16}$; *coefficienti* 4 *denominatoris* a *autem subscribatur denominator quantitatis defectivæ, ut habeatur defectus* $\frac{4}{20}$. *Sit denique ratio excessiva, ut* $4 : 1\frac{3}{4} = 16 : 7$, *& defectiva ut* $4 : \frac{6}{7} = 28 : 6$: *erunt lunulæ* $\frac{7}{16}$ *&* $\frac{6}{28} = \frac{196}{448}$ *&* $\frac{96}{448}$, *consequenter* $\frac{196}{445} - \frac{96}{448} = \frac{100}{448}$ *summa excessus & defectus, cujus numerator est* $= 3d + a = 84 + 16$. *Iam cum coefficiens denominatoris* d *sit* 3; *subscribatur ei denominator quantitatis excessivæ, ut habeatur excessus* $\frac{3}{16}$. *Quoniam vero denominator* a *caret coefficiente, subscribatur unitati denominator quantitatis defectivæ, & habebitur defectus* $\frac{1}{28}$. *Cum itaque ablatis excessibus ex lunulis excessivis, vel additis defectibus ad defectivas, hic quoque prodeat lunula vera* $= \frac{1}{4}$; *palam est, excessus & defectus per problema præcedens exacte posse determinari.*

§. 18. Scholion II. *Sit diametri quadratum* $= 1$, *ratio ejus ad segmentum excessiva, ut* $64 : 9\frac{2}{5} = 320 : 47$ *& defectiva ut* $64 : 8\frac{1}{4} = 256 : 33$: *erunt segmenta* $\frac{47}{320}$ *&* $\frac{33}{256} = \frac{12032}{81920}$ *&* $\frac{10560}{81920}$: *consequenter* $\frac{12032}{81920} - \frac{10560}{81920} = \frac{1472}{81920}$ *summa excessus & defectus, cujus numerator est* $= 2d + 3a = 512 + 960$. *Ergo (per* §. 16.) *excessus est* $\frac{2}{320}$ *& defectus* $\frac{3}{256}$: *consequenter segmentum verum* $\frac{47}{320} - \frac{2}{320} = \frac{45}{320} = \frac{9}{64}$ *vel* $\frac{33}{256} + \frac{3}{256} = \frac{36}{256} = \frac{9}{64}$. *Sit deinde ratio excessiva, ut* $64 : 9\frac{1}{2} = 128 : 19$, *& defectiva, ut* $64 : 8\frac{1}{3} = 192 : 25$: *erunt segmenta* $\frac{19}{128}$ *&* $\frac{25}{192} = \frac{3648}{24576}$ *&* $\frac{3200}{24576}$: *consequenter* $\frac{3548}{24576} - \frac{3200}{24576} = \frac{448}{24576}$ *summa excessus & defectus, cujus numerator est* $= d + 2a = 192 + 256$. *Ergo excessus est* $\frac{1}{128}$ *& defectus* $\frac{2}{192}$: *consequenter segmentum verum* $\frac{19}{128} - \frac{1}{128} = \frac{18}{128} = \frac{9}{64}$; *vel* $\frac{25}{192} + \frac{2}{192} = \frac{27}{192} = \frac{9}{64}$. *Sit denique ratio excessiva* $64 : 9\frac{2}{3} = 192 : 29$ *& defectiva* $64 : 8\frac{1}{2} = 128 : 17$: *erunt segmenta* $\frac{29}{192}$ *&* $\frac{17}{128} = \frac{3712}{24576}$ *&* $\frac{3264}{24576}$, *consequenter* $\frac{3712}{24576} - \frac{3264}{24576} = \frac{448}{24576}$ *summa excessus & defectus, cujus numerator est* $= 2d + a = 256 + 192$. *Ergo excessus est* $\frac{2}{192}$ *& defectus* $\frac{1}{128}$: *consequenter segmentum verum* $\frac{29}{192} - \frac{2}{192} = \frac{27}{192} = \frac{9}{64}$; *vel* $\frac{17}{128} + \frac{1}{128} = \frac{18}{128} = \frac{9}{64}$. *Ergo segmentum est ad quadratum diametri ut* $\frac{9}{64} : 1. = 9 : 64$, *consequenter diameter ad peripheriam, ut* $8 : 25$ (§. 8.).

§. 19. Scholion III. *Sit diameter* $= 1$, *ratio ejus ad peripheriam excessiva, ut* $8 : 25\frac{2}{3} = 24 : 77$, *& defectiva, ut* $8 : 24\frac{1}{5} = 40 : 121$: *erunt peripheriæ per utramque rationem inventæ* $\frac{77}{24}$ *&* $\frac{121}{40} = \frac{3080}{960}$ *&* $\frac{2904}{960}$: *consequenter* $\frac{3080}{960} - \frac{2904}{960} = \frac{176}{960}$ *summa excessus & defectus, cujus numerator est* $= 2d + 4a = 80 + 96$. *Ergo (per* §. 16.) *excessus est* $\frac{2}{24}$ *& defectus* $\frac{4}{40}$: *consequenter peripheria vera* $\frac{77}{24} - \frac{2}{24} = \frac{75}{24} = \frac{25}{8}$; *vel* $\frac{121}{40} + \frac{4}{40} = \frac{125}{40} = \frac{25}{8}$. *Sit deinde ratio excessiva,*

ſiva, *ut* $8:26$ *& defectiva*, *ut* $8:24\frac{1}{9} = 72:217$: *erunt peripheriæ* $\frac{26}{8}$ *&* $\frac{217}{72} = \frac{1872}{576}$ *&* $\frac{1736}{576}$, *conſequenter* $\frac{1872}{576} - \frac{1735}{576} = \frac{136}{576}$ *ſumma exceſſus & defectus*, *cujus numerator eſt* $= d \dagger 8a = 72 \dagger 64$. *Ergo exceſſus eſt* $\frac{1}{8}$ *& defectus* $\frac{8}{72}$, *conſequenter peripheria vera* $\frac{26}{8} - \frac{1}{8} = \frac{25}{8}$; *vel* $\frac{217}{72} \dagger \frac{8}{72} = \frac{225}{72} = \frac{25}{8}$. *Sit denique ratio exceſſiva, ut* $8:25\frac{5}{7} = 56:181$, *& defectiva*, *ut* $8:24$; *erunt peripheriæ* $\frac{181}{56}$ *&* $\frac{24}{8} = \frac{1448}{448}$ *&* $\frac{1344}{448}$, *conſequenter* $\frac{1448}{448} - \frac{1344}{448} = \frac{104}{448}$ *ſumma exceſſus & defectus*, *cujus numerator eſt* $= 6d \dagger a = 48 \dagger 56$ *Ergo exceſſus eſt* $\frac{6}{56}$ *& defectus* $\frac{1}{8}$: *conſequenter peripheria vera* $\frac{175}{56} = \frac{25}{8}$; *vel* $\frac{24}{8} \dagger \frac{1}{8} = \frac{25}{8}$. *Eſt itaque diameter ad peripheriam*, *ut* $1 : \frac{25}{8} = 8:25$.

§. 20. *Corollarium.* Cum igitur per infinitas rationes numerator ſummæ exceſſus & defectus evadat aggregatum ex denominatoribus quantitatum exceſſivæ & defectivæ 1mo ſimplis, deinde multiplis, ac tandem ex multiplo denominatoris unius atque ex ſimplo alterius, & in quovis caſu, ablato exceſſu ex quantitate exceſſiva vel addito defectu ad defectivam, conſtanter prodeat eadem ratio, nempe: quadrati diametri ad lunulam ut $4:1$; & ad ſegmentum, ut $64:9$; & ratio diametri ad peripheriam, ut $8:25$; evidens eſt quamlibet ex ratiocinio legitimo & ex principiis inconcuſſis, non autem ex combinatione arbitraria numerorum, eſſe deductam.

THEOREMA.

§. 21. *Peripheriæ* Ludolphina, Metiana & Archimedea *peccant in exceſſu*: *1ma* $\frac{3}{200}$, *2da* $\frac{15}{904}$ *& 3tia* $\frac{9}{568}$ *diametri.*

Demonſtratio. Quoniam ratio per quemcunque numerum 3tium multiplicata manet ſemper eadem; multiplicetur ratio *Ludolphi à Ceulen* $100:314$ per 2, quo facto oritur $= 200:628$. Aſſumtis deinde diametro $= 1$ & ratione ejus ad peripheriam defectiva $8:24$, prodeunt peripheriæ $\frac{628}{200}$ & $\frac{24}{8} = \frac{5024}{1600}$ & $\frac{4800}{1600}$, conſequenter $\frac{5024}{1600} - \frac{4800}{1600} = \frac{224}{1600}$ ſumma exceſſus & defectus (§. 2.), cujus numerator eſt $= 3d \dagger a = 24 \dagger 200$. Ergo (per §. 16.) exceſſus eſt $\frac{3}{200}$ & defectus $\frac{1}{8}$. *Quod erat 1mum.*

Ratio *Adriani Metii* $113:355$ multiplicata per 8, dat $= 904:2840$, quæ cum defectiva $8:24$ efficit peripherias $\frac{2840}{904}$ & $\frac{24}{8} = \frac{22720}{7232}$ & $\frac{21696}{7232}$, conſequenter $\frac{22720}{7232} - \frac{21696}{7232} = \frac{1024}{7232}$ ſumma exceſſus & defectus, cujus numerator eſt $= 15d \dagger a = 120 \dagger 904$. Ergo exceſſus eſt $\frac{15}{904}$ & defectus $\frac{1}{8}$. *Quod erat ſecundum.*

Ratio *Archimedis* $1:3\frac{10}{71}$ multiplicata per 568 prodit $= 568:1784$, quæ cum defectiva $8:24$ manifeſtat peripheras $\frac{1784}{568}$ & $\frac{24}{8} = \frac{14272}{4544}$ & $\frac{13632}{4544}$, conſequenter $\frac{14272}{4544} - \frac{13632}{4544} = \frac{640}{4544}$ ſumma exceſſus & defectus, cujus numerator eſt $= 9d \dagger a = 72 \dagger 568$. Ergo exceſſus eſt $\frac{9}{568}$ & defectus $\frac{1}{8}$. *Quod erat 3tium.*